Muon g-2 in the standard model and a lattice QCD calculation of the leading hadronic contribution

Laurent Lellouch

CPT & IPhU Marseille
CNRS & Aix-Marseille U.

Budapest-Marseille-Wuppertal collaboration [BMWc]

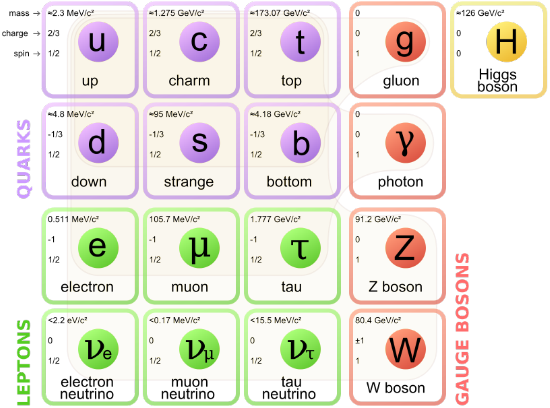
Borsanyi, Fodor, Guenther, Hoelbling, Katz, LL, Lippert, Miura, Szabo, Parato, Stokes, Toth, Torok, Varnhorst

Nature 593 (2021) 51, online 7 April 2021 $\rightarrow$ BMWc '20
PRL 121 (2018) 022002 (Editors' Selection) $\rightarrow$ BMWc '17
& Aoyama et al., Phys. Rept. 887 (2020) 1-166 $\rightarrow$ WP '20

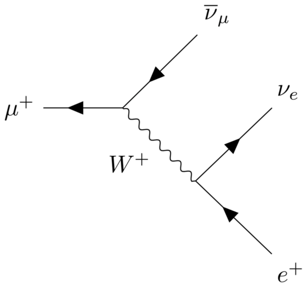
The Standard Model and beyond

The Standard Model on a page

Relativistic quantum field theory that describes all known elementary particles and three of the four fundamental interactions



(Wikimedia)



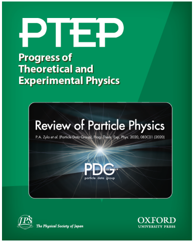
- muon (μ) $\sim$ electron (e): same interactions w/ gauge bosons, not with the Higgs

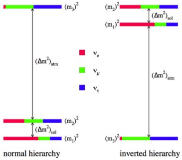
$\rightarrow$ $m_\mu \simeq 207 \times m_e$ & $\tau_\mu \simeq 2 \times 10^{-6}$ sec

Why go beyond the Standard Model?

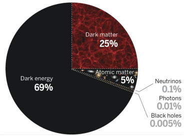
SM is an incredibly successful theory: since mid 70's it has been tested against experiment thousands of times and has never failed

Particle Data Group's "Review of Particle Physics": $\sim$ 2100 pp. of measurements, almost all explained/explainable by SM





(Wikipedia)

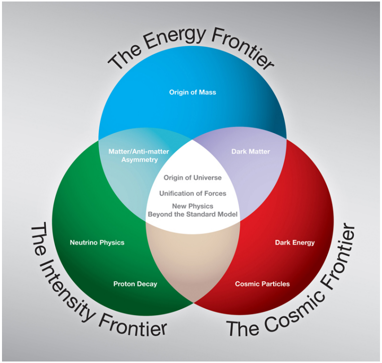


(D.N. Spergel, Science '15)

However, SM leaves important questions unanswered:

- Why three families of matter particles?
- How do neutrinos acquire mass?
- Can the 26 parameters needed to describe elementary particles be predicted?
- Is the Higgs mechanism all there is to electroweak symmetry breaking?
- How to include gravity?
- Why do we see more matter than antimatter in the universe?
- What is dark matter?
- Why is the expansion of the universe accelerating?
- . . .

Searching for new fundamental physics



- **Cosmic frontier:** use the universe as an observatory to learn about particles physics

  $\rightarrow$ e.g. is dark matter a new elementary particle?

- **Energy frontier:** particle beams are collided at the highest possible energies to directly produce new particles and phenomena

  $\rightarrow$ e.g. is the Higgs whose properties are measured at the LHC really just the SM Higgs?

- **Intensity frontier:** high-flux beams and/or high-precision, low-energy experiments are used to indirectly uncover new particles or forces in effects of minute quantum fluctuations

  $\rightarrow$ e.g. does the measurement of the magnetic moment of the muon harbor physics beyond the SM?

Strategy: measure observable as precisely as possible and compute SM prediction w/ commensurate precision

measurement $=$ SM prediction ?

If not, then new fundamental physics

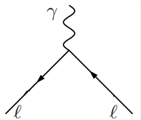
$\Leftarrow$

Lepton magnetic moments and motivation

Leptons in magnetic fields: early history of electron

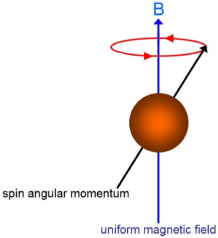
A massive particle w/ electric charge and spin behaves like a tiny magnet in a magnetic field

The Dirac eqn (1928) predicts that a lepton ℓ has magnetic moment



$$\vec{\mu}_\ell = g_\ell \left(\frac{e_\ell}{2m_\ell}\right)\vec{S}, \qquad \vec{S} = \hbar\frac{\vec{\sigma}}{2}$$

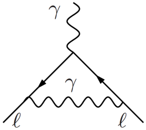
$$g_\ell\big|_{\text{Dirac}} = 2$$



"That was really an unexpected bonus for me" (P.A.M. Dirac)

- In 1934, Kinsler & Houston confirmed g_e=2 to $\sim$ 0.1% w/ Zeeman effect in neon
- However in 1947, Nafe, Nels & Rabi observe a deviation of g_e=2 in hyperfine structure of hydrogen and deuterium, then measured precisely by Kusch & Foley

  $\rightarrow$ deviation at 0.1% level

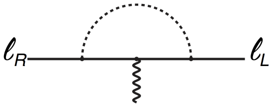


Schwinger (1947) immediately understands that effect comes from quantum, particle fluctuations in the vacuum and computes

$$a_e \equiv \frac{g_e - 2}{2} = \frac{\alpha}{2\pi} = 0.0116\cdots$$

$\Rightarrow$ birth of QED and relativistic quantum field theory

Why so excited about the muon magnetic moment?



$\longrightarrow \quad \frac{a_\ell}{2m_\ell} eF^{\mu\nu}[\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$

- Actually interested in $a_\ell = (g_\ell - 2)/2$, $\ell = e, \mu$: finite to all orders in renormalizable theories and measured, very precisely $\Rightarrow$ excellent tests of SM and BSM theories
- Loop induced $\Rightarrow$ sensitive to dofs that may be too heavy or too weakly coupled to be produced directly
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to: EDMs, *s* and *b* decays, LHC direct searches, ...
- As early as 1956, Berestetskii noted that sensitivity of a_ℓ to contributions of heavy particles w/ $M \gg m_\ell$ typically goes like $\sim (m_\ell/M)^2$

  $\Rightarrow a_\mu$ is $(m_\mu/m_e)^2 \sim 43,000$ times more sensitive to heavy dofs than a_e

  $\Rightarrow a_\mu$ sensitive to possibly unknown, heavy dofs
- Despite $\tau_\mu \sim 2\,\mu$s, a_μ measured in 1960 [Garwin et al '60]

  $\rightarrow$ measurements progressed in // with the development of the SM, each new experiment probing theory to a new level
- Early 2000s, BNL measured a_μ to 0.54 ppm: EW contribution seen at 3σ level

  $\rightarrow$ But also excess over SM prediction $\sim 2\times$ EW contribution

Why so excited about the muon magnetic moment?

- Since then, persistent tension between measurement & SM $> 3.5\sigma$
- To decide on possible presence of BSM physics:
  - significant upgrade of BNL experiment @ FNAL w/ goal to reduce measurement error by factor of 4
  - important theoretical effort to improve SM prediction to same level

⇒ White Paper from the muon $g-2$ Theory Initiative posted on arXiv in June 2020 w/ reference SM prediction [Aoyama et al '20 = WP '20]

⇒ Presentation and publication on April 7 of FNAL's first results (only 6% of planned data)
$\rightarrow$ tour de force measurement confirms BNL result w/ already improved precision
$\rightarrow$ reduces WA error to 0.35 ppm and increases tension w/ SM to 4.2σ

- Same day, *Nature* published our *ab-initio* calculation of hadronic vacuum polarization contribution to the SM prediction that brings it much closer to measurement of a_μ

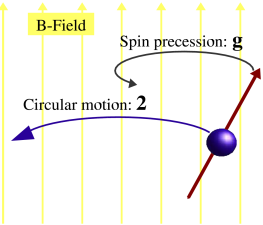
Big question:

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}}?$$

If not, there must be new Φ

Experimental measurement of a_μ

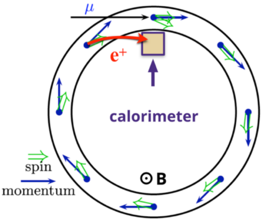
Measurement principle for a_μ



Precession determined by

$$\vec{\mu}_\mu = 2(1+a_\mu)\frac{Qe}{2m_\mu}\vec{S}$$

$$\vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c}\vec{S}$$



$$\vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta = -\frac{Qe}{m_\mu}\left[a_\mu \vec{B} + \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] - \eta_\mu \frac{Qe}{2m_\mu}\left[\frac{\vec{E}}{c} + \vec{\beta}\times\vec{B}\right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$ &

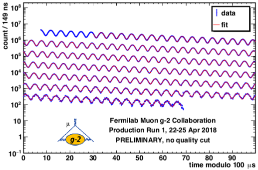
$$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

since $d_\mu = 0.1(9)\times 10^{-19} e\cdot\mathrm{cm}$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E}=0$ (J-PARC)

$$\to \Delta\omega \simeq -a_\mu B \frac{Qe}{m_\mu}$$

Fermilab E989 @ magic γ: measurement (simplified)



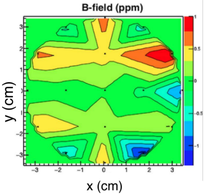
ω_a

Extract from decay positron time spectra

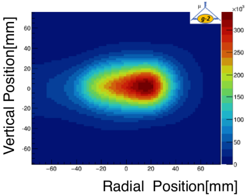
$$N(t) = N_0 e^{-t/\tau_\mu}[1 + A\cos(\omega_a t + \phi)]$$

$$a_\mu = \left(\frac{g_e}{2}\right)\left(\frac{\omega_a}{\langle\omega_p\rangle}\right)\left(\frac{\mu_p}{\mu_e}\right)\left(\frac{m_\mu}{m_e}\right)$$

0.26 ppt — 3 ppb — 22 ppb ⇨ 2017 CODATA



Map the magnetic field



Obtain muon distribution In the storage ring

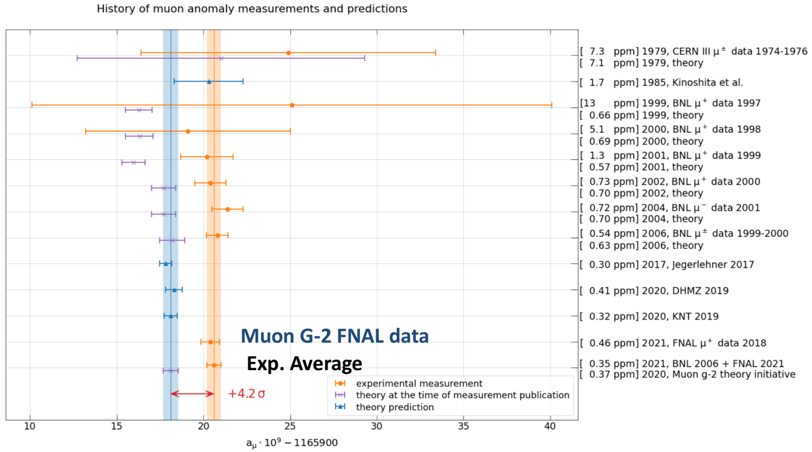
$$\langle\omega_p\rangle \approx \omega_p \otimes \rho(r)$$

Average magnetic field weighted by muon distribution

ω_p: free proton precession frequency

Using proton NMR $\hbar\omega_p = 2\mu_p B$

$g_\mu - 2$ updated history (7 April 2021)



$$a_\mu(\text{AVG}) = 116\,592\,061(41) \times 10^{-11} \quad (0.35\,\text{ppm}).$$

71

G. Venanzoni, CERN Seminar, 8 April 2021

Bathroom scale sensitive to the weight of a single eyelash !!!

Based on only 6% of expected FNAL data! $\rightarrow$ aim $\delta a_\mu = 0.14\,\text{ppm}$

Standard model calculation of a_μ

At needed precision: all three interactions and all SM particles

$$\begin{aligned}
a_\mu^{\text{SM}} &= a_\mu^{\text{QED}} + a_\mu^{\text{had}} + a_\mu^{\text{EW}} \\
&= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{e}{4\pi\sin\theta_W}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}$$

QED contributions to a_ℓ

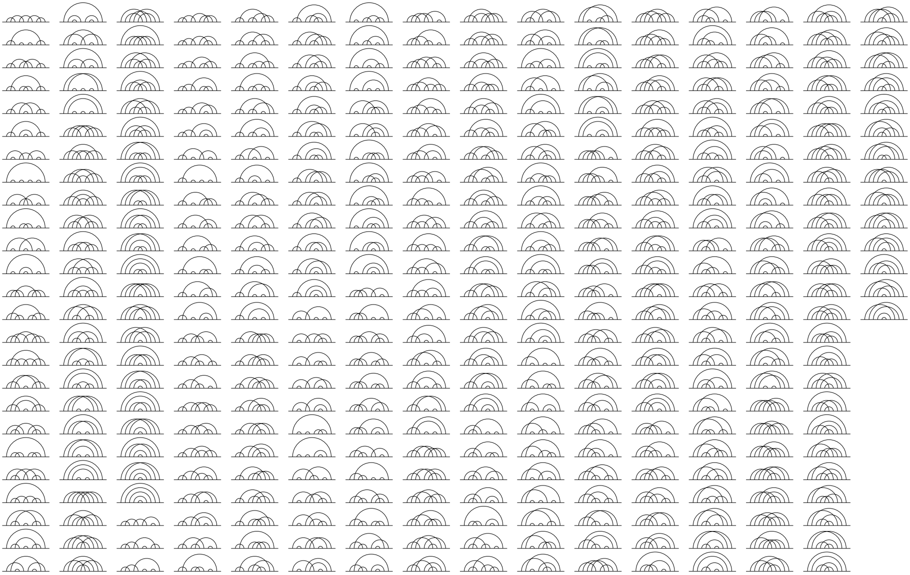
Loops with only photons and leptons

$$a_\ell^{\text{QED}} = C_\ell^{(2)}\left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

$$C_\ell^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\ell/m_{\ell'}) + A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$$

- $A_1^{(2)}, A_1^{(4)}, A_1^{(6)}, A_2^{(4)}, A_2^{(6)}, A_3^{(6)}$ known analytically (Schwinger '48; Sommerfield '57, '58; Petermann '57; . . .)
- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al '91, '93, '95, '96; Kinoshita '95)
- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta '95; Aguilar et al '08; Aoyama, Kinoshita, Nio '96-'18)
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Only some diagrams are known analytically
  - Not all contributions are fully, independently checked

5-loop QED diagrams



(Aoyama et al '15)

QED contribution to a_μ

$$a_\mu^{\text{QED}}(Cs) = 1\,165\,847\,189.31(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(23)_{\alpha(Cs)} \times 10^{-12} \text{ [0.9 ppb]}$$

$$a_\mu^{\text{QED}}(a_e) = 1\,165\,847\,188.42(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(28)_{\alpha(a_e)} \times 10^{-12} \text{[0.9 ppb]}$$

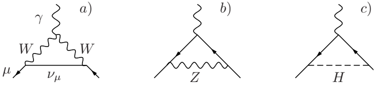
(Aoyama et al '12, '18, '19)

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} = 734.2(4.1) \times 10^{-10}$$

$$\stackrel{?}{=} a_\mu^{\text{EW}} + a_\mu^{\text{had}}$$

Electroweak contributions to a_μ: Z, W, H, etc. loops

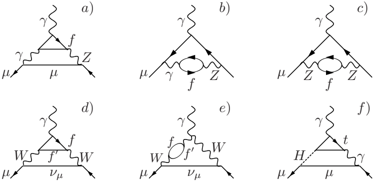
1-loop



$$a_\mu^{\mathrm{EW},(1)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\right)$$
$$= 19.479(1) \times 10^{-10}$$

(Gnendiger et al '15, Aoyama et al '20 and refs therein)

2-loop



$$a_\mu^{\mathrm{EW},(2)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right)$$
$$= -4.12(10) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_\mu^{\mathrm{EW}} = 15.36(10) \times 10^{-10}$$

Hadronic contributions to a_μ: quark and gluon loops

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 718.9(4.1) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

Clearly right order of magnitude:

$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

(already Gourdin & de Rafael '69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

Huge challenge: theory of strong interaction between quarks and gluons, QCD, hugely nonlinear at energies relevant for a_μ

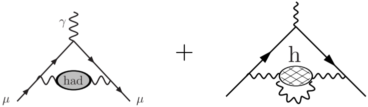
$\to$ perturbative methods used for electromagnetic and weak interactions do not work

$\to$ need nonperturbative approaches

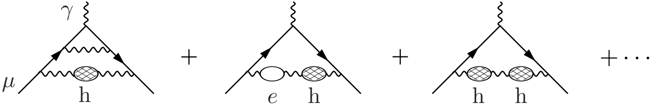
Write

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

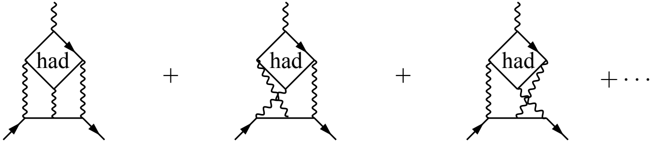
Hadronic contributions to a_μ: diagrams



$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$

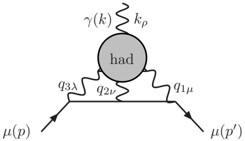


$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

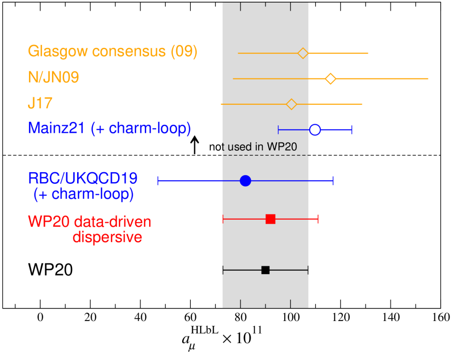


$\rightarrow a_\mu^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

Hadronic light-by-light

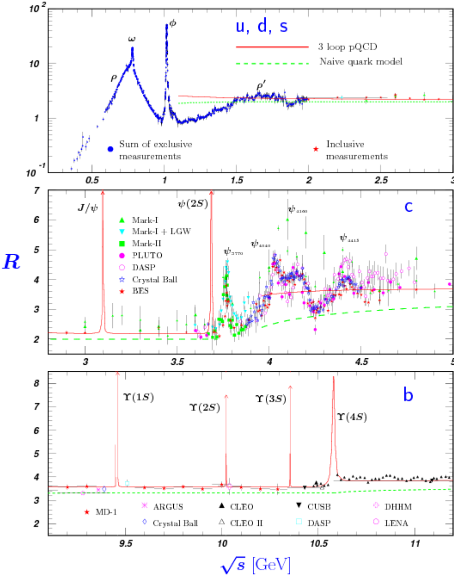


- HLbL much more complicated than HVP, but ultimate precision needed is $\simeq$ 10% instead of $\simeq$ 0.2%
- For many years, only accessible to models of QCD w/ difficult to estimate systematics (Prades et al '09): $a_\mu^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$
- Also, lattice QCD calculations were exploratory and incomplete
- Tremendous progress in past 5 years:
  - $\rightarrow$ Phenomenology: rigorous data driven approach [Colangelo, Hoferichter, Kubis, Procura, Stoffer,. . . '15-'20]
  - $\rightarrow$ Lattice: first two solid lattice calculations
- All agree w/ older model results but error estimate much more solid and will improve
- Agreed upon average w/ NLO HLbL and conservative error estimates [WP '20]
- $a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} - a_\mu^{\text{HLbL}} = 709.7(4.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{HVP}}$



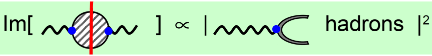
[Colangelo '21]

HVP from $e^+e^- \to$ had (or $\tau \to \nu_\tau +$ had)



(PDG compilation)

Use [Bouchiat et al 61] optical theorem (unitarity)



$$\mathrm{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+e^- \to \text{had})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

and a once subtracted dispersion relation (analyticity)

$$\begin{aligned} \hat{\Pi}(Q^2) &= \int_0^\infty ds\, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s) \\ &= \frac{Q^2}{12\pi^2} \int_0^\infty ds\, \frac{1}{s(s+Q^2)} R(s) \end{aligned}$$

$\Rightarrow \hat{\Pi}(Q^2)$ & $a_\mu^{\text{LO-HVP}}$ from data: sum of exclusive $\pi^+\pi^-$ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.

$$a_\mu^{\text{LO-HVP}} = 694.0(1.0)(3.9) \times 10^{-10} \ [0.6\%]$$

[DHMZ'19] (sys. domin.)

Can also use $I(J^{PC}) = 1(1^{--})$ part of $\tau \to \nu_\tau +$ had and isospin symmetry + corrections

Standard model prediction and comparison to experiment

SM prediction vs experiment on April 7, 2021 (v1)

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
HVP LO (R-ratio)	692.8 ± 2.4	[KNT '19]
	694.0 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
HVP LO (R-ratio, avg)	693.1 ± 4.0	[WP '20]
HVP LO (lattice$<$2021)	711.6 ± 18.4	[WP '20]
HVP NLO	-9.83 ± 0.07	[Kurz et al '14, Jegerlehner '16, WP '20]
HVP NNLO	1.24 ± 0.01	[Kurz '14, Jeger. '16]
HLbyL LO (pheno)	9.2 ± 1.9	[WP '20]
HLbyL LO (lattice$<$2021)	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
HLbyL LO (lattice 2021)	$10.7 \pm 1.1 \pm 0.9$	[Mainz '21]
HLbyL LO (avg)	9.0 ± 1.7	[WP '20]
HLbyL NLO (pheno)	0.2 ± 0.1	[WP '20]
QED [5 loops]	11658471.8931 ± 0.0104	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[Gnendiger '15, WP '20]
HVP Tot. (R-ratio)	684.5 ± 4.0	[WP '20]
HLbL Tot.	9.2 ± 1.8	[WP '20]
SM [0.37 ppm]	11659181.0 ± 4.3	[WP '20]
Exp [0.35 ppm]	11659206.1 ± 4.1	[BNL '06 + FNAL '21]
Exp $-$ SM	25.1 ± 5.9 [4.2σ]	

SM prediction vs experiment on April 7, 2021 (v2)

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
HVP LO (R-ratio)	692.8 ± 2.4	[KNT '19]
	694.0 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
HVP LO (R-ratio, avg)	693.1 ± 4.0	[WP '20]
HVP LO (lattice)	707.5 ± 5.5	[BMWc '20]
HVP NLO	-9.83 ± 0.07	
		[Kurz et al '14, Jegerlehner '16, WP '20]
HVP NNLO	1.24 ± 0.01	[Kurz '14, Jeger. '16]
HLbyL LO (pheno)	9.2 ± 1.9	[WP '20]
HLbyL LO (lattice<2021)	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
HLbyL LO (lattice 2021)	$10.7 \pm 1.1 \pm 0.9$	[Mainz '21]
HLbyL LO (avg)	9.0 ± 1.7	[WP '20]
HLbyL NLO (pheno)	0.2 ± 0.1	[WP '20]
QED [5 loops]	11658471.8931 ± 0.0104	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[Gnendiger '15, WP '20]
HVP Tot. (lat. + R-ratio)	698.9 ± 5.5	[WP '20, BMWc '20]
HLbL Tot.	9.2 ± 1.8	[WP '20]
SM [0.49 ppm]	11659195.4 ± 5.7	[WP '20 + BMWc '20]
Exp [0.35 ppm]	11659206.1 ± 4.1	[BNL '06 + FNAL '21]
Exp − SM	10.7 ± 7.0 [1.5σ]	

Very brief introduction to lattice QCD

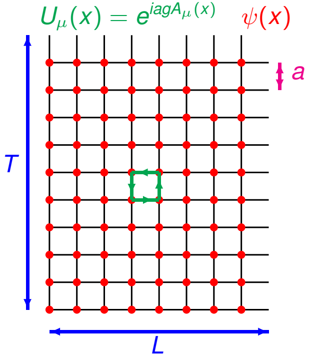
What is lattice QCD (LQCD)?

To describe matter w/ sub-% precision, QCD requires $\geq$ 104 numbers at every spacetime point
$\rightarrow \infty$ number of numbers in our continuous spacetime
$\rightarrow$ must temporarily “simplify” the theory to be able to calculate *(regularization)*
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\langle O \rangle = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G - \int \bar{\psi} D[M] \psi}\, O[U, \psi, \bar{\psi}]$$
$$= \int \mathcal{D}U\, e^{-S_G}\, \mathrm{det}(D[M])\, O[U]_{\mathrm{Wick}}$$

- $\mathcal{D}Ue^{-S_G}\,\mathrm{det}(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\mathrm{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^9)$ dofs) challenge

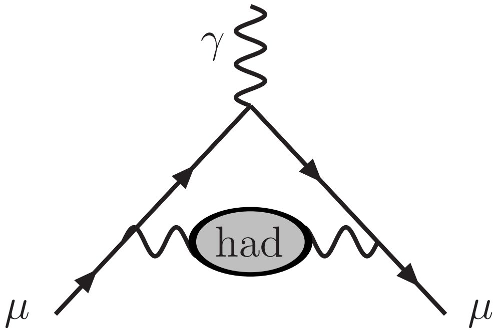
Our “accelerators”

Such computations require some of the world’s most powerful supercomputers

- 1 year on supercomputer
  $\sim$ 100 000 years on laptop

- In Germany, those of the Forschungszentrum Jülich, the Leibniz Supercomputing Centre (Munich), and the High Performance Computing Center (Stuttgart); in France, Turing and Jean Zay at the Institute for Development and Resources in Intensive Scientific Computing (IDRIS) of the CNRS, and Joliot-Curie at the Very Large Computing Centre (TGCC) of the CEA, by way of the French Large-scale Computing Infrastructure (GENCI).

Lattice QCD calculation of a_μ^{HVP}



All quantities related to a_μ will be given in units of 10^{-10}

HVP from LQCD: introduction

Consider in Euclidean spacetime, i.e. spacelike $q^2 = -Q^2 \leq 0$ [Blum '02]

$$\Pi_{\mu\nu}(Q) =$$

$$= \int d^4x\, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Then [Lautrup et al '69, Blum '02]

$$a_\ell^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{m_\ell^2}\, k(Q^2/m_\ell^2) \hat{\Pi}(Q^2)$$

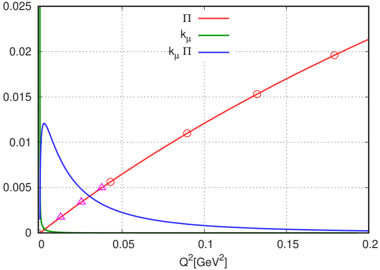
w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$

FV & $a \neq 0$: discrete momenta, $\Pi_{\mu\nu}(0) \neq 0$ & $\Pi(0) \sim \ln a$

$\rightarrow$ modify Fourier transform to take care of all three problems and eliminate some noise [Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...]

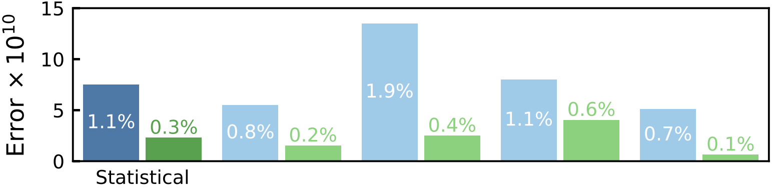
Contributions of ud, s, c ... have very different systematics (and statistical errors) on lattice

$\rightarrow$ study each one individually

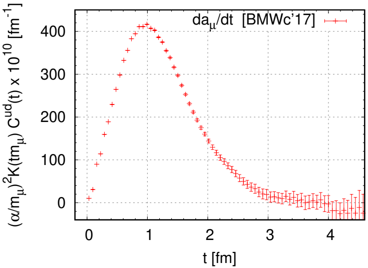


$(k_\mu(Q^2) = (\pi/m_\mu)^2 k(Q^2))$

Key improvements: statistical noise reduction

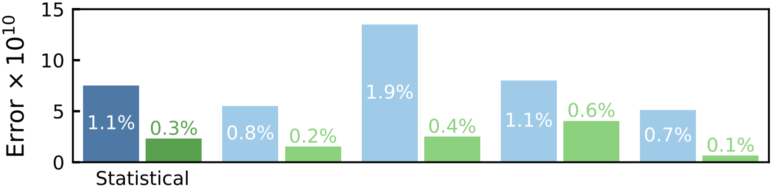


Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP "bubble"



(144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

Key improvements: statistical noise reduction

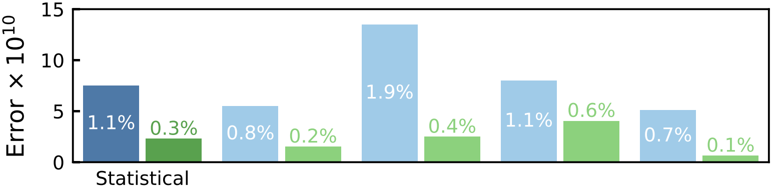


Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP "bubble"

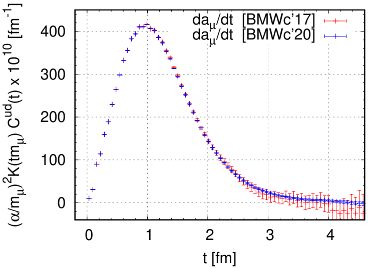
Solve w/:

- Algorithmic improvements (EigCG, solver truncation [Bali et al '09], all mode averaging [Blum et al '13]) to generate more statistics: $> 25,000$ gauge configurations & tens of millions of measurements
- Exact treatment of long-distance modes to reduce long-distance noise (low mode averaging [Neff et al '01, Giusti et al '04, ...])
- Rigorous upper/lower bounds on long-distance contribution [Lehner '16, BMWc '17]

Key improvements: statistical noise reduction

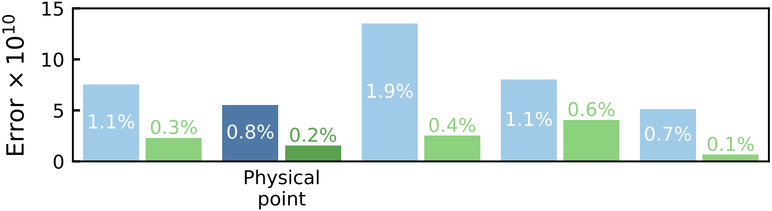


Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP "bubble"



(144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

Key improvements: tuning of QCD parameters

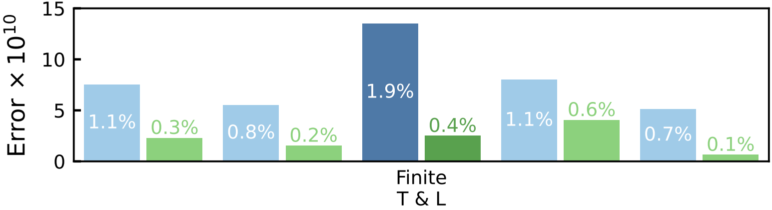


Must tune parameters of QCD very precisely: m_u, m_d, m_s, m_c & overall mass scale

Solve w/:

- Permil determination of overall QCD scale
- Set w/ Ω^- baryon mass computed w/ 0.2% uncertainty
- Use Wilson flow scale [Lüscher '10, BMWc '12] to separate out electromagnetic corrections

Key improvements: remove finite spacetime distortions

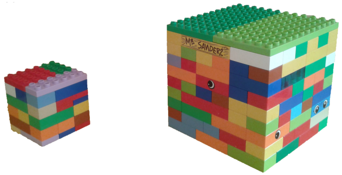


Even on "large" lattices ($L \gtrsim 6\,\mathrm{fm}$, $T \gtrsim 9\,\mathrm{fm}$), early pen-and-paper estimate [Aubin et al '16] suggested that exponentially suppressed finite-volume distortions are still $O(2\%)$

Solve by:

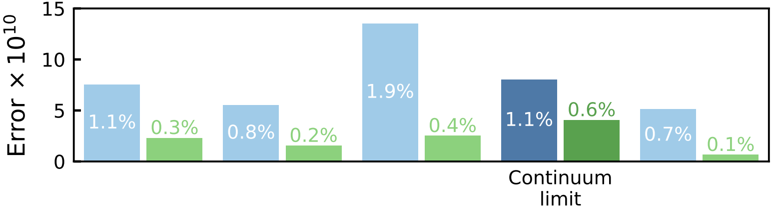
- Finding a way to perform dedicated supercomputer simulations to calculate effect between above and much larger $L = T = 11\,\mathrm{fm}$ volume directly in QCD, i.e. "big" − "ref"
- Computing remnant $\sim 0.1\%$ effect in "big" volume w/ simplified models of QCD that correctly predict "big" − "ref"

"ref" →

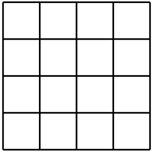


← "big"

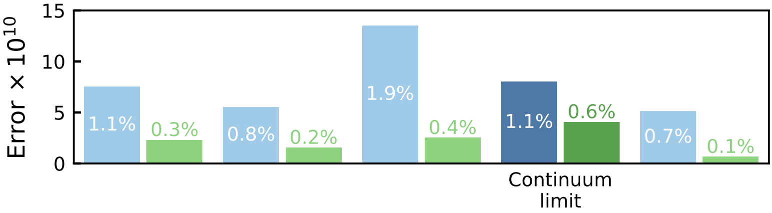
Key improvements: controlled continuum limit



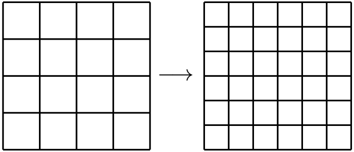
Our world corresponds to spacetime w/ lattice spacing $a \to 0$



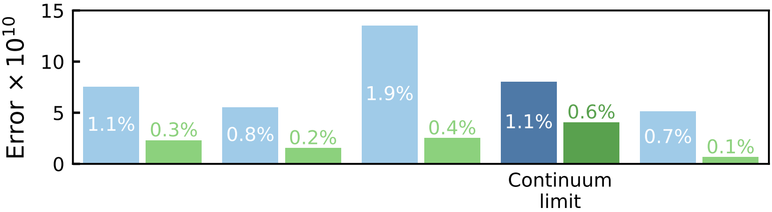
Key improvements: controlled continuum limit



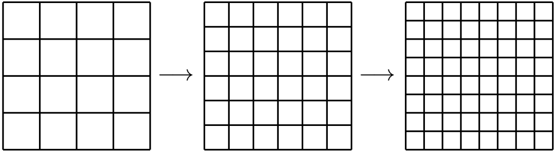
Our world corresponds to spacetime w/ lattice spacing $a \to 0$



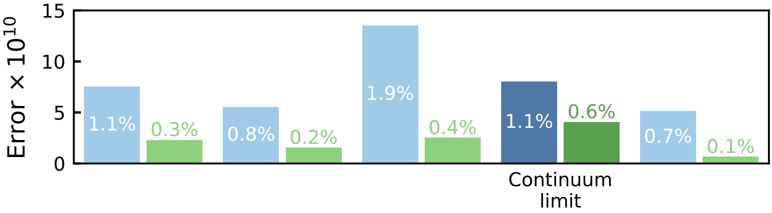
Key improvements: controlled continuum limit



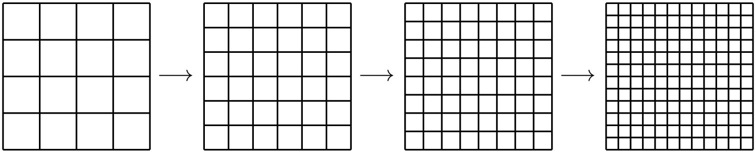
Our world corresponds to spacetime w/ lattice spacing $a \to 0$



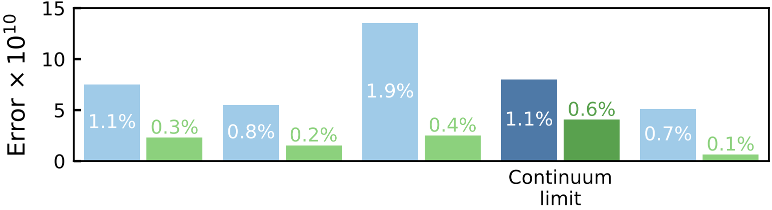
Key improvements: controlled continuum limit



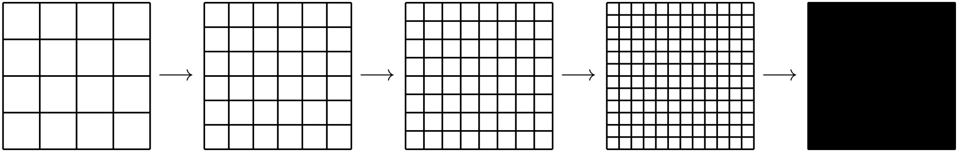
Our world corresponds to spacetime w/ lattice spacing $a \to 0$



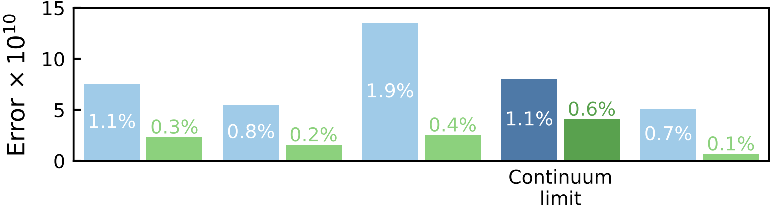
Key improvements: controlled continuum limit



Our world corresponds to spacetime w/ lattice spacing $a \to 0$



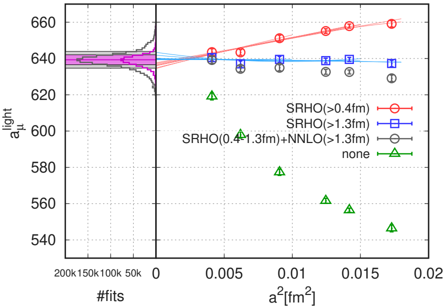
Key improvements: controlled continuum limit



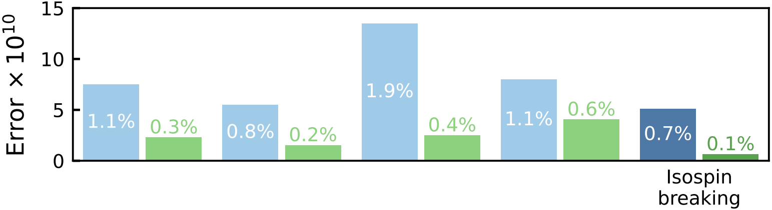
Our world corresponds to spacetime w/ lattice spacing $a \to 0$

Control $a \to 0$ extrapolation of results by:

- Performing all calculations on lattices w/ 6 values of a in range $0.134\,\mathrm{fm} \to 0.064\,\mathrm{fm}$
- Reducing statistical error at smallest a from 1.9% to 0.3% !
- Improving approach to continuum limit w/ simplified models for QCD [Sakurai '60, Bijnens et al '99, Jegerlehner et al '11, Chakraborty et al '17, BMWc '20] shown to reproduce distortions observed at $a>0$
- Extrapolate results to $a=0$ using theory as guide



Key improvements: QED and $m_u \neq m_d$ corrections



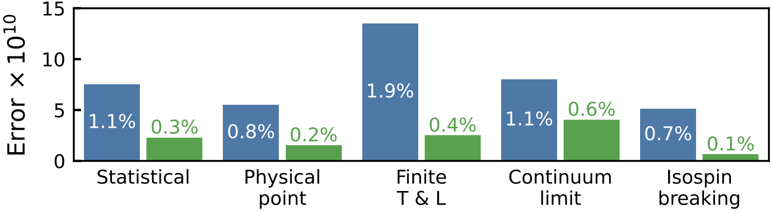
For subpercent accuracy, must include small effects from electromagnetism and due to fact that masses of *u* and *d* quarks are not quite equal

- Effects are proportional to powers of $\alpha = \frac{e^2}{4\pi} \sim 0.01$ and $\frac{m_d - m_u}{(M_p/3)} \sim 0.01$

$\Rightarrow$ for SM calculation at permil accuracy sufficient to take into account contributions proportional to only first power of α or $\frac{m_d - m_u}{(M_p/3)}$

- We include *all* such contributions for *all* calculated quantities needed in calculation

Robust determination of uncertainties



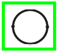
Thorough and robust determination of statistical and systematic uncertainties

- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach [BMWc '08, '14]
  - Hundreds of thousands of different analyses of correlation functions
  - Weighted by AIC weight
  - Use median of distribution for central values & $16 \div 84\%$ confidence interval to get total error

(Nature paper has 95 pp. Supplementary information detailing methods)

Summary of contributions to $a_\mu^{\text{LO-HVP}}$

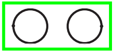
Isospin symmetric



connected light
633.7(2.1)(4.2)

connected strange
53.393(89)(68)

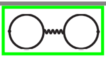
connected charm
14.6(0)(1)

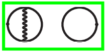


disconnected
-13.36(1.18)(1.36)

QED isospin-breaking: valence

connected -1.23(40)(31)

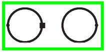




disconnected -0.55(15)(10)

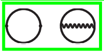
Strong isospin-breaking

connected
6.60(63)(53)

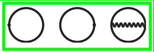


disconnected
-4.67(54)(69)

QED isospin-breaking: sea



connected 0.37(21)(24)



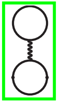
disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed



connected -0.0093(86)(95)

disconnected 0.011(24)(14)

Finite-size effects

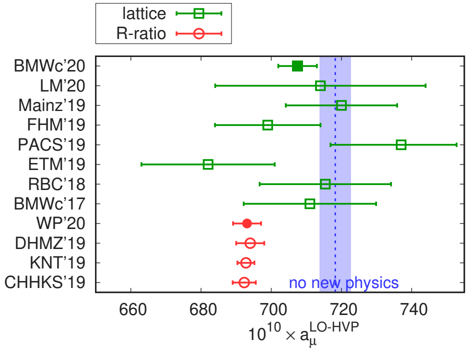
isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

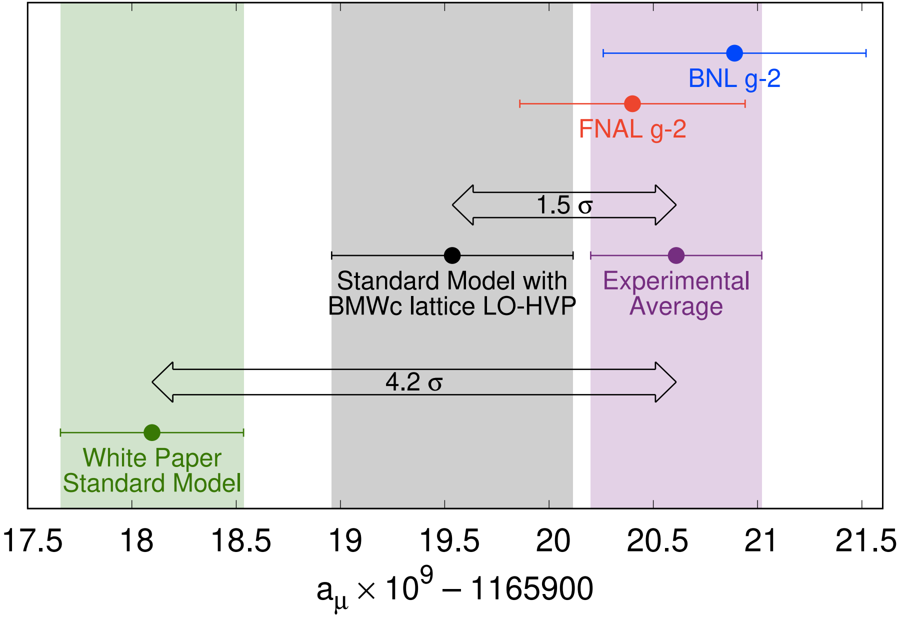
Comparison and outlook

Comparison



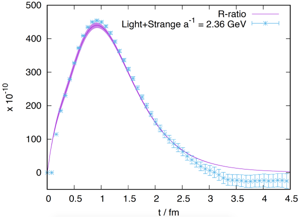
- Consistent with other lattice results
- Total uncertainty is divided by $3 \div 4$ . . .
- . . . and comparable to R-ratio and experiment
- Consistent w/ experiment @ 1.5σ ("no new physics" scenario) !
- 2.1σ larger than R-ratio average value [WP '20]

Fermilab plot, April 7 2021, BMWc version

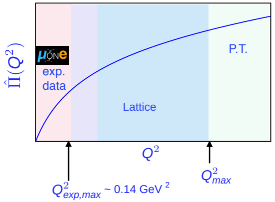


What next?

- FNAL to reduce WA error by factor of $2 \div 3$ in coming years
- HLbL error must be reduced by factor of $1.5 \div 2$
- Must reduce ours by factor of 4 !
- And must reduce proportion of *systematics* in theory error
- Will experiment still agree with our prediction ?
- Must be confirmed by other lattice groups
- If confirmed, must understand why lattice doesn't agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty [RBC/UKQCD '18]
- Important to pursue $e^+e^- \to$ hadrons measurements [BaBar, CMD-3, BES III, Belle II, ...]
- $\mu e \to \mu e$ experiment MUonE very important for experimental crosscheck and complementarity w/ LQCD
- Important to build J-PARC $g_\mu - 2$ and pursue a_e experiments



[RBC/UKQCD '18]



[Marinkovic et al '19]