On 't Hooft lines and Lax operators of SO_{2N} type

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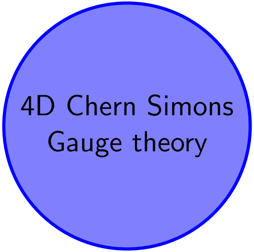
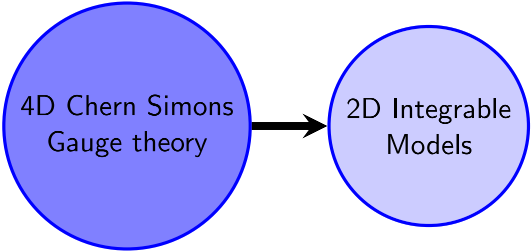
Overview


4D Chern Simons
Gauge theory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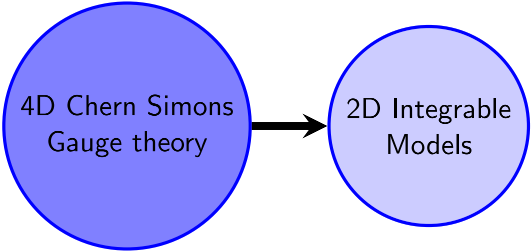
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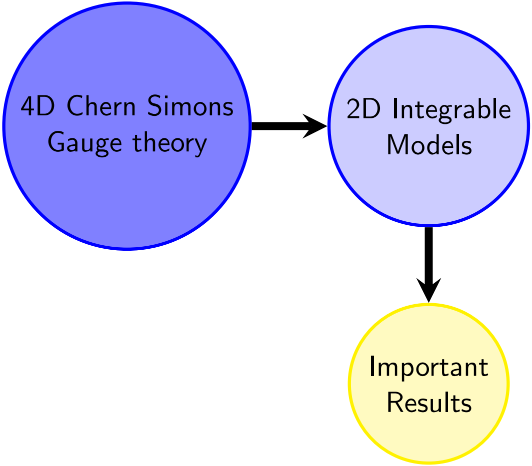
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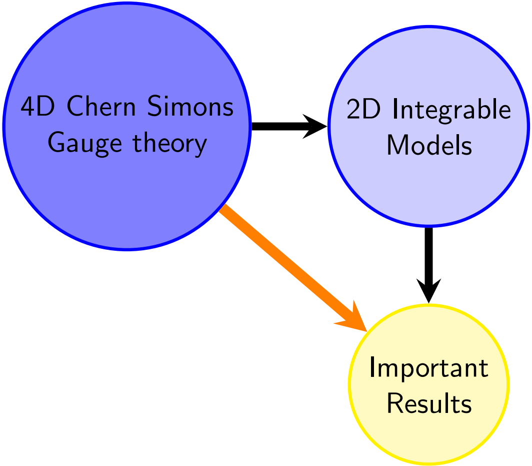
Important Results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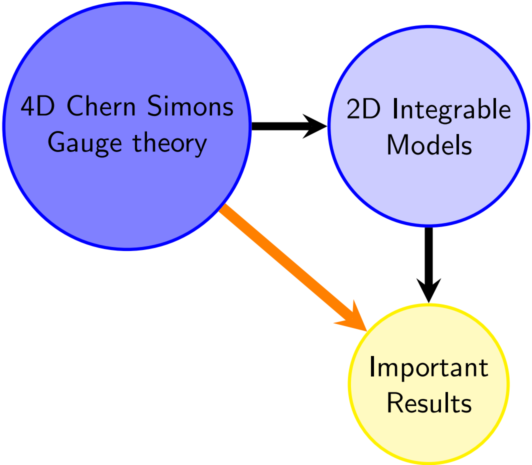
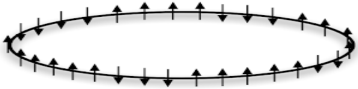


Goal **:** Derive Lax operators for spin chains of D-type from 4D CS theory

- Heisenberg XXX spin chain



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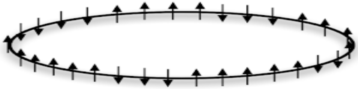

- Yang Baxter equation

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2)$$

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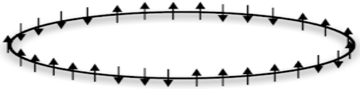

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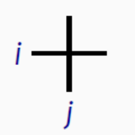
- R-matrix : $R_{ij}(z_i - z_j) : V_i \otimes V_j \rightarrow V_i \otimes V_j$
  $|i >\in V_i$, $|j >\in V_j \rightarrow$ vector spaces states



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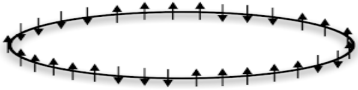

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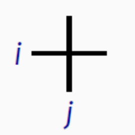
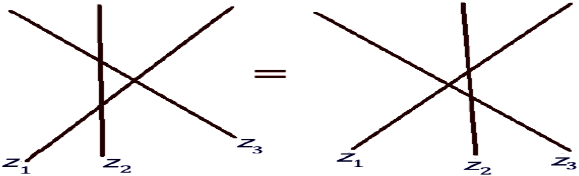
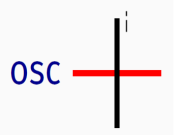


- Graphical YBE



- Lax matrix : $L_i = R_{i,aux} : V_i \to V_i$
  $|i> \in V_i$ vector space
  auxiliary oscillator space : $[a, a^+] = 1$



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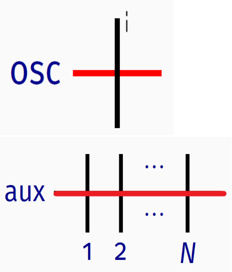

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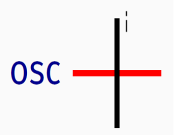

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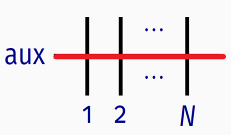

- RLL equations

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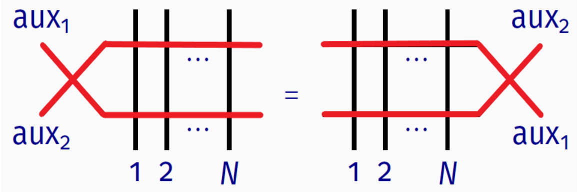

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- Equation of motion

$$F = dA + A \wedge A = 0$$

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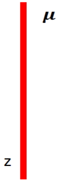
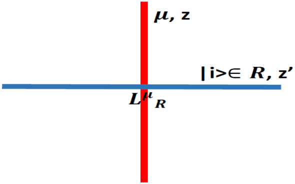
- 't Hooft line defects
  -Curve in $\mathbb{R}^2$, in a point $z \in \mathbb{C}$
  Magnetic charge = coweight of G



Gauge/Integrability Correspondence

- Line defect in $\mathbb{R}^2$ $\rightarrow$ worldline of a particle in moving spacetime
- Quantum space V $\rightarrow$ representation R of G.
- Point z in $\mathbb{C}$ $\rightarrow$ spectral parameter of the particle
- Wilson line $\rightarrow$ electrically charged particle
- 't Hooft line $\rightarrow$ magnetically charged particle, Dirac monopole
- Crossing of two Wilson lines $\rightarrow$ interaction between two particles

- Lax operator = Crossing of a Wilson line with a 't Hooft line



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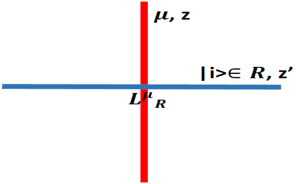

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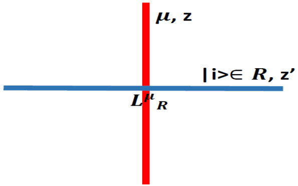
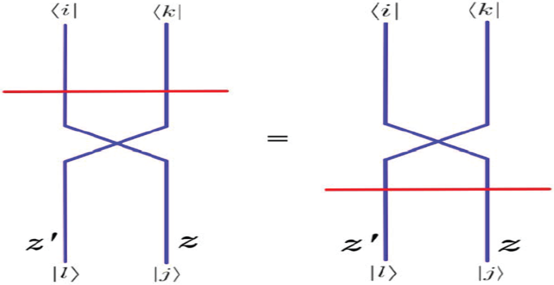


- = Parallel transport of gauge fields from below to above the 't Hooft line.
- The RLL equations $R^{ik}_{rs}(z-w)L^r_j(z)L^s_l(w) = L^i_r(w)L^k_s(z)R^{rs}_{jl}(z-w)$



- Minuscule coweight μ of G
  $\rightarrow$ Eigenvalues of μ on g are $0, \pm 1$

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  $\rightarrow$ Minuscule Lax operator for G:

$$L = e^{X} z^{\mu} e^{Y}$$

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- Levi constraints

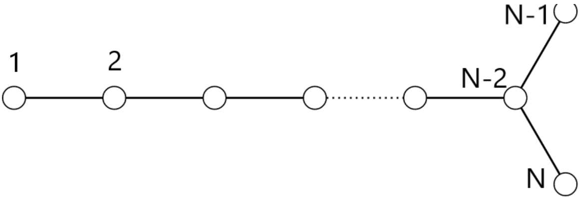
$$[\mu, X_\alpha] = +X_\alpha$$

$$[\mu, Y^\alpha] = -Y^\alpha$$

- Oscillator realisation :

$$[b^\alpha, c_\beta] = \delta^\alpha_\beta$$

- $G = SO_{2N}$:

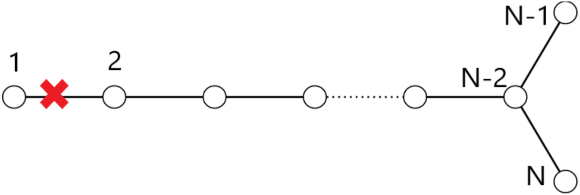


$$\alpha_i = e_i - e_{i+1}, 1 \leq i \leq N-2; \alpha_{N-1} = e_{N-1} - e_N; \alpha_N = e_{N-1} + e_N$$

- $\rightarrow$ 3 minuscule coweights: μ_1, μ_{N-1}, μ_N
  $\rightarrow$ 3 minuscule Lax operators: 1 vector + 2 spinorial :

$$L_D^{\mu_1}, L_D^{\mu_{N-1}}, L_D^{\mu_N}$$

- Vector coweight: $\mu_1 = e_1$



- Levi decomposition :

$$so_{2N} \to \underbrace{so_2 \oplus so_{2N-2}}_{l_\mu} \oplus \underbrace{(2N-2)_\pm}_{n_\pm}$$

- Fundamental representation decomposition

$$2N \to 1_+ \oplus (2N-2)_0 \oplus 1_-$$

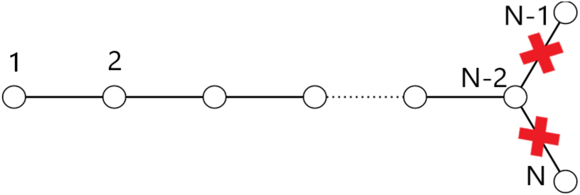
- Vector minuscule Lax operator of SO_{2N} :
  In the basis: $|+\rangle, |i\rangle, |-\rangle, 1 \leq i \leq N-2$:
  $\mu = |+\rangle\langle+| + 0|i\rangle\langle i| - |-\rangle\langle-|$
  $X_i = |+\rangle\langle i| - |i\rangle\langle-|$
  $Y^i = |i\rangle\langle+| - |-\rangle\langle i|$

$$L_D^{\mu_1} = \begin{pmatrix} z^2 + zbc + \frac{1}{4}b^2c^2 & zb - \frac{1}{2}b^2c & \frac{1}{2}b^2 \\ zc - \frac{1}{2}bc^2 & z\mathbb{I}_{2N-2} + bc & -b \\ \frac{1}{2}c^2 & -c & 1 \end{pmatrix}$$

- 2N-2 couples of oscillators $\mathbf{b} = (b^1, ..., b^{2N-2})$ and $\mathbf{c} = (c_1, ..., c_{2N-2})$

$$[b^i, c_j] = \delta^i_j$$

- Spinor coweights: $\mu_{N-1} = e_1 + ... - e_N$ and $\mu_N = e_1 + ... + e_N$



- Levi decomposition :

$$so_{2N} \to \underbrace{so_2 \oplus sl_N}_{l_\mu} \oplus \underbrace{N(N-1)_\pm}_{n_\pm}$$

- Fundamental representation decomposition

$$2N \to N_{+\frac{1}{2}} \oplus N_{-\frac{1}{2}}$$

- Spinor minuscule Lax operator of SO_{2N} :
  In the basis: $|i\rangle, |\bar{i}\rangle, 1 \leq i \leq N, \bar{i} = 2N + 1 - i$:
  $\mu = |i\rangle| + -|\bar{i}\rangle\langle\bar{i}|$
  $X_{[ij]} = |i\rangle\langle\bar{j}| - |j\rangle\langle\bar{i}|$
  $Y^{[ij]} = |\bar{i}\rangle\langle j| - |\bar{j}\rangle\langle i|$

$$L_D^{\mu_N} = \begin{pmatrix} \mathbb{I}_N + BC & B \\ C & \mathbb{I}_N \end{pmatrix}$$

$$B = \begin{cases} b^{i\bar{j}}, i < j \\ -^{j\bar{i}}, j < i \\ 0, i = j \end{cases} \qquad B = \begin{cases} c_{\bar{j}i}, i < j \\ -c_{\bar{i}j}, j < i \\ 0, i = j \end{cases}$$

- N(N-1) couples of oscillators $b^{[ij]}$ and $c_{[ij]}$

$$[b^{ij}, c_{kl}] = \delta^{ij}_{kl}$$

Conclusions and Perspectives

- Results obtained from 4D CS theory agree with results of the spin chains literature
- Results for other symmetry groups and other representations
- Interpretations of results from new points of view
- Linking to supersymmetric 6D $N = 1$ Yang Mills theory
- Linking to string theories (M-branes, type II strings...)

Thanks for your attention