On the quantum geometry of gravity

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Motivation

- Analogy of the algebraic deformation of classical mechanics to quantum mechanics
- Describe algebraically the gravity

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Definition (Algebra)

An algebra$(H, +, \cdot, \eta; \mathbb{K})$ over $\mathbb{K}$ is a ring $(H, +, \cdot)$ and an action of $\mathbb{K}$ which is compatible with both the product and addition. $(H, +, \mathbb{K})$ is a vector space and the product is $\cdot : H \otimes H \to H$. The unit axiom is expressed as for $h \in H$ $\eta_h : \mathbb{K} \to H$ such that $\eta_h(\lambda) = \lambda h$ for $\lambda \in \mathbb{K}$ with $\eta_{1_H} = \eta$

The associativity of the product and the unit element are expressed as the following commutative diagrams

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$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{id \otimes \cdot} & H \otimes H \\ {\scriptstyle \cdot \otimes id}\big\downarrow & & \big\downarrow{\scriptstyle \cdot} \\ H \otimes H & \xrightarrow{\quad \cdot \quad} & H \end{array} \qquad \begin{array}{ccccc} H \otimes \mathbb{K} & \xrightarrow{id \otimes \eta} & H \otimes H & \xleftarrow{\eta \otimes id} & \mathbb{K} \otimes H \\ & {\scriptstyle \cong}\searrow & \big\downarrow{\scriptstyle \cdot} & \swarrow{\scriptstyle \cong} & \\ & & H & & \end{array} \tag{1}$$

Definition (Coalgebra)

A *Coalgebra* $(C,+,\Delta,\varepsilon;\mathbb{K})$ is a vector space $(C,+,\mathbb{K})$ with a coproduct $\Delta : C \to C \otimes C$ which is coassociative and for which there exist a counit $\varepsilon : C \to \mathbb{K}$. For $c \in C$ we will write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$$

The axioms of coassociativity and the counit element are expressed in the following commutative diagrams

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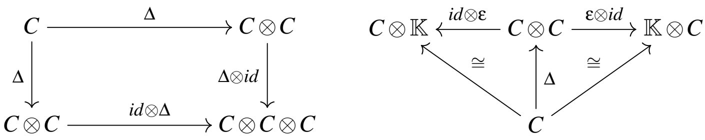

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Definition (Bialgebra)

A *Bialgebra* $(H,+,.,\eta,\Delta,\varepsilon;\mathbb{K})$ is a vector space $(H,+,\mathbb{K})$ for which $(H,+,.,\eta;\mathbb{K})$ is an algebra and $(H,\Delta,\varepsilon;\mathbb{K})$ is a coalgebra. With Δ and ε are an algebra map, such that

$$\Delta(hg) = \Delta(h)\Delta(g) \qquad \varepsilon(hg) = \varepsilon(h)\varepsilon(g)$$
$$\Delta(1) = 1 \otimes 1 \qquad \varepsilon(1) = 1$$

Definition (Hopf Algebra)

A *Hopf algebra* $(H,+,.,\eta,\Delta,\varepsilon;\mathbb{K})$ is a bialgebra over $\mathbb{K}$ equipped with the antipode $S : H \to H$ defined such that

$$\cdot(S \otimes id) \circ \Delta = \cdot(id \otimes S) \circ \Delta = \eta\varepsilon$$

To the following presentation, we denote by H a bialgebra or Hopf algebra.

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Duality

The axioms of Hopf algebra expressed in (1) and (2) are self-dual. We denote by H^* the dual of a Hopf algebra H built on the vector space dual to H. We can express the Hopf structure on h^* as follows

$$\begin{aligned} \langle \phi\psi, h\rangle &= \langle \phi\otimes\psi, \Delta h\rangle \\ \langle \Delta\phi, h\otimes g\rangle &= \langle \phi, hg\rangle \\ \langle S\phi, h\rangle &= \langle \phi, Sh\rangle \end{aligned} \qquad \begin{aligned} \langle 1, h\rangle &= \varepsilon(h) \\ \varepsilon(\phi) &= \langle \phi, 1\rangle \end{aligned}$$

Paired Hopf Algebra

Two Hopf algebra A and H are paired if there is a bilinear map $\langle\ ,\ \rangle : A\otimes H \to \mathbb{K}$ obeying to the above equations for $\phi, \psi \in A$ and $h, g \in H$. They are strictly dual pair if the pairing is non degenerate. A such pairing can be made by quotienting out those elements that pair as zero with all the elements of the other Hopf algebra.

Action on a vector space

The left action of H on a vector space V is a pair $(V,\triangleright)$ with a linear map $\triangleright : H \otimes V \to V$ defined as

$$h \triangleright v \in V \qquad (hg) \triangleright v = h \triangleright (g \triangleright v)$$
$$1 \triangleright v = v$$

For $h \in H$ we can also write $\rho(h) = h \triangleright ()$ with $\rho : H \to Lin(V)$. Sometimes we denote the action by α such that $\alpha(h \otimes v) = \alpha_h(v)$ and $\rho(h) = \alpha_h()$

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It is more interesting to say that, here, V is a left H-module.

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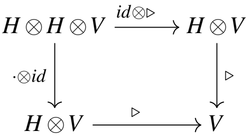

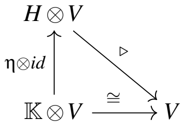

It is more interesting to say that, here, V is a left H-module.

Action on algebras

Let A an algebra and H a Hopf algebra, A is a H-module if A is a left H-module with $\triangleright$ an algebra map and

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \qquad h \triangleright 1 = \varepsilon(h)1$$

for $a, b \in A$ and $h \in H$

Action on coalgebras

A coalgebra C is a left H-module coalgebra if

$$\Delta(h \triangleright c) = h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleright c_{(2)} \qquad \varepsilon(h \triangleright c) = \varepsilon(h)\varepsilon(c)$$

for $c \in C$ and $h \in H$. $\triangleright : H \otimes C \to C$ is a coalgebra map and $\Delta(h \triangleright c) = \Delta(h) \triangleright \Delta(c)$

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Coaction of coalgebra

It is a pair (V, β) with V a vector space and $\beta : V \to V \otimes H$ such that

$$(\beta \otimes id) \circ \beta = (id \otimes \Delta) \circ \beta \qquad id = (id \otimes \varepsilon) \circ \beta$$

In terms of comodule, we have

$$v^{(1)(1)} \otimes v^{(1)(2)} \otimes v^{(2)} = v^1 \otimes v^{(2)}_{(1)} \otimes v^{(2)}_{(2)}$$

$$v^{(1)}\varepsilon(v^{(2)}) = v$$

Proposition (cross product)

Let H be a Hopf algebra and let A be a left H-module algebra. There is a left cross product algebra $A \rtimes H$ built on $A \otimes H$ with product

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b) \otimes h_{(2)} g$$

The unit element is $1 \otimes 1$

Proposition (cross coproduct)

Let H be a Hopf algebra and let C be a left H-comodule coalgebra. There is a right cross coproduct algebra $H \ltimes C$ built on $H \otimes C$ with the coalgebra structure

$$\Delta(h \otimes c) = h_{(1)} \otimes c_{(1)}^{(1)} h_{(2)} c_{(1)}^{(2)} \otimes c_{(2)} \qquad \varepsilon(h \otimes c) = \varepsilon(h)\varepsilon(c)$$

for $h \in H, c \in C$

Construction of Bicrossed product

Let A and H two Hopf algebras, let A a left H-module algebra and H a right comodule coalgebra

$$\alpha : H \otimes A \to A \qquad h \otimes a \mapsto h \triangleright a$$

$$\beta : H \to H \otimes A \qquad h \mapsto h^{(1)} h^{(2)}$$

obeying to

$$\varepsilon(h \triangleright a) = \varepsilon(h)\varepsilon(a)$$

$$\beta(1) = 1 \otimes 1$$

$$h^{(1)}_{(2)} \otimes (h_{(1)} \triangleright a) h^{(2)}_{(2)} = h^{(1)}_{(1)} \otimes h^{(2)}_{(1)} (h_{(2)} \triangleright a)$$

$$\Delta(h \triangleright a) = h^{(1)}_{(1)} \triangleright a_{(1)} \otimes h^{(2)}_{(1)} (h_{(2)} \triangleright a_{(2)})$$

$$\beta(gh) = g^{(1)}_{(1)} h^{(1)} \otimes g^{(2)}_{(1)} (g_{(2)} \triangleright h^{(2)})$$

Then $A \rtimes H$ and $A \ltimes H$ form a bialgebra: the left-right bicrossproduct bialgebra associated to the compatible (co)actions and denoted by $A \blacktriangleright\!\!\triangleleft H$ with the antipode

$$S(a \otimes h) = (1 \otimes Sh^{(1)})(S(ah^2) \otimes 1)$$

Heisenberg algebra

It is a three dimensional Lie algebra generated by $a, a^\dagger$ and H with the following commutation relation

$$[a, a^\dagger] = 0 \qquad \begin{aligned} [H, a] &= 0 \\ [H, a^\dagger] &= 0 \end{aligned} \tag{3}$$

We can define a coproduct on it such that

$$\begin{aligned} \Delta a &= a \otimes 1 + 1 \otimes a \\ \Delta a^\dagger &= a^\dagger \otimes 1 + 1 \otimes a^\dagger \\ \Delta H &= H \otimes 1 + 1 \otimes H \end{aligned}$$

Quantization of phase space

Let consider a particle moving on homogeneous space (G, M, α). M, a smooth manifold, describes the position space and G, a Lie group, the momentum group. The right action of G is such that

$$\alpha : M \times G \to M$$

For $\xi \in g$(Lie algebra) generate a flow in M of the form

$$\begin{aligned} s(t) &= \alpha_{e^{t\xi}}(s(0)) \\ &= e^{t\xi} \triangleright (s(0)) \end{aligned}$$

$s(t) \in M,\ t \in \mathbb{R}$. For G semi-simple and H reductive, there is a metric on M in which its geodesic is described by those flows.

The particle motion needs to refer to a point of M, and one could work with $C^\infty(M)$: if a particle is at $s \in M$, one could specify $f(s) \in C^\infty(M)$. Geodesics are labeled by g which is the classical momentum observables. We can work too with $X = T^*M$ such that $C^\infty(X)$ is the classical algebra of observables with $g \otimes C^\infty(M) \subseteq C^\infty(X)$. This inclusion is the consequence of the pull-back of projection $T^*M \to M$ and $\alpha_* : g \to vect(M)$ The vector fields induced by the action is

$$(\alpha_*\xi)(f)(s) = \frac{d}{dt}\mid_{t=0} f(\alpha_{e^{t\xi}}(s)) \qquad \forall f \in C^\infty(M)$$

Quantization step

The quantum algebra should be some *-subalgebra of operators on a Hilbert space and contain the quantum counterpart of the classical observables of interest. For that we a map $\widehat{\ }$ that we call *quantization map* described as follows:

$$[\widehat{\xi},\widehat{\eta}] = \widehat{[\xi,\eta]} \qquad \forall \xi,\eta \in g$$
$$\widehat{f}\widehat{g} = \widehat{hg} \qquad \forall f,g \in C^\infty(M)$$

We require the commutation relation (3) such that

$$[\widehat{\xi},\widehat{f}] = \widehat{(\alpha_*\xi)(f)} \qquad\qquad \alpha_*\xi = o(\hbar)$$

To require a bounded operators, we will work with the group G instead of g, so the Heisenberg commutation relation in the group form is

$$e^{t\widehat{\xi}}\widehat{f}e^{\widehat{-t\xi}} = \widehat{f \triangleleft e^{t\xi}}$$
$$\alpha_{e^{t\xi}}(f)(s) = f(\alpha_{e^{t\xi}}(s))$$

A quantum system don't have any definite position so we could generalize G to be a locally compact instead of Lie group and M as locally compact space. So we will work with $\mathbb{K}(M)$ (set of function on M with values in $\mathbb{K}$) to identify the space and $\mathbb{K}G$ (vector space with basis G) which are both form a Hopf algebra. Our problem is to find A such that

$$\mathbb{K}(M) \xrightarrow{\;\widehat{}\;} A \xleftarrow{\;\widehat{}\;} \mathbb{K}G$$

$$\widehat{u}\widehat{f}\widehat{u^{-1}} = \widehat{\alpha_u(f)} \qquad \forall u \in G, f \in \mathbb{K}(M)$$

Universal solution of the quantization problem

$\mathbb{K}(M) \rtimes \mathbb{K}G$ with canonical inclusion of $\mathbb{K}G$ and $\mathbb{K}(M)$ is a solution of the quantization problem

If there is another solution A, there is a unique algebra map

$$\phi_A : \mathbb{K}(M) \rtimes \mathbb{K}G \to A$$
$$1 \otimes u \mapsto \widehat{u}$$
$$f \otimes 1 \mapsto \widehat{f}$$

$\forall u \in G, f \in \mathbb{K}(M)$

This is a Bialgebra and not a Hopf algebra. To construct a Hopf algebra, we use the bicrossed product

discussion

The quantum algebra of observable is a non commutative version of the algebra of functions $C(X)$ where X is the classical phase space, the degree of non commutativity is controlled by $\hbar$.

$$\mathbb{K}(M) \bowtie \mathbb{K}G \xrightarrow{\hbar \to 0} C(X)$$

If $C(X)$ is a Hopf algebra, there exist a coproduct that makes it correspond to a group structure. The non commutativity of the coproduct correspond to non-Abelian group structure on the phase space which give us a curvature on that space. In the frame of general relativity, the degree of curvature is controlled by G(the gravitational constant) so

$$\mathbb{K}(M) \bowtie \mathbb{K}G \xrightarrow{G \to 0} \mathbb{K}(M) \rtimes \mathbb{K}G$$

where $\mathbb{K}(M) \rtimes \mathbb{K}G$ is the usual quantum mechanics

Summary

- The bicrossed product allow us to construct a Hopf algebra which describe the phase space
- By the Hopf duality, we have unified in one algebraic structure the macroscopic geometry and quantum mechanics
- what if we use a relativistic frame to obtain a field theory?