Extra-dimensional theory & phenomenology in the era of gravitational waves

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arXiv:2211.08489 [gr-qc]
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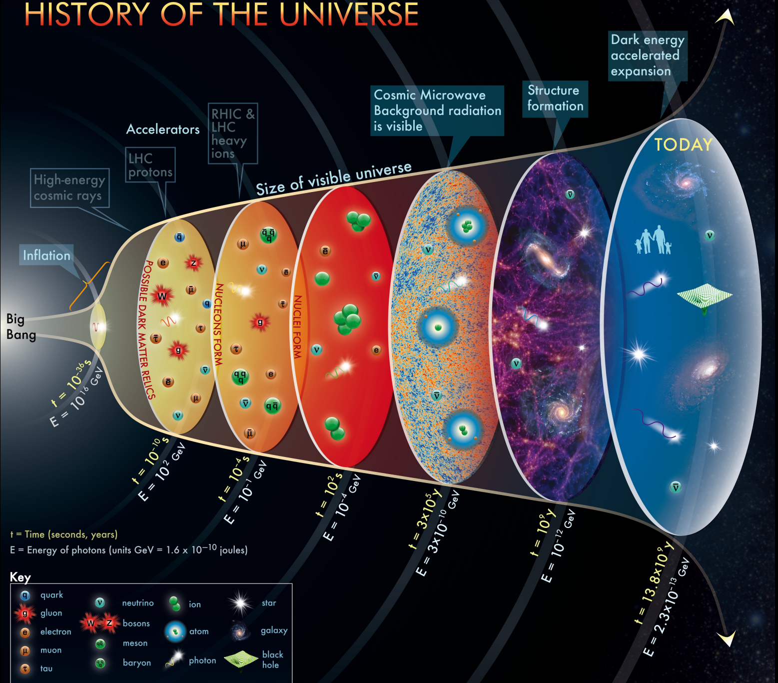
Outline

HISTORY OF THE UNIVERSE



t = Time (seconds, years)

E = Energy of photons (units GeV = 1.6 x 10^{-10} joules)

Key

quark	neutrino	ion	star
gluon	W Z bosons	atom	galaxy
electron	meson	photon	black hole
muon	baryon		
tau			

The concept for the above figure originated in a 1986 paper by Michael Turner.

Supported by DOE

Explore beyond the CMB and CνB...

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- Phase transitions: QCD ($\sim$ 100 MeV), EW ($\sim$100 GeV)

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- Inflation ($\leq 10^{16}$ GeV)

The allure of GWs

Explore beyond the CMB and $C\nu B$...

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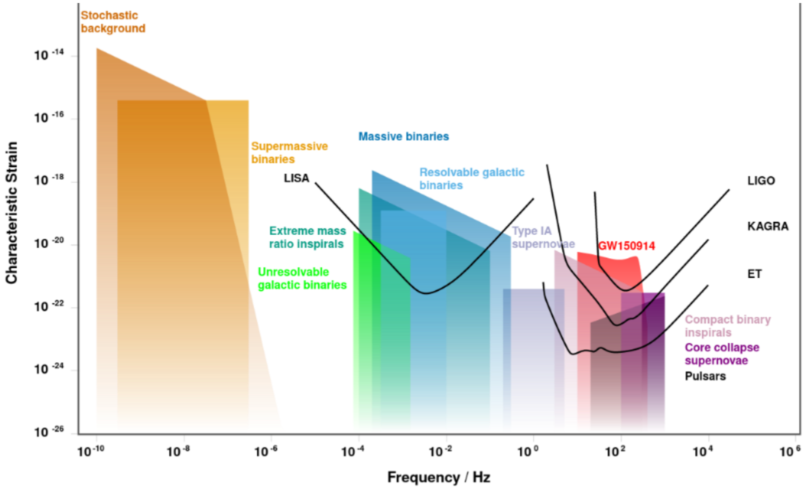
- Phase transitions: QCD ($\sim$ 100 MeV), EW ($\sim$100 GeV)

  $\hookrightarrow$ Baryogenesis + baryon asymmetry, EWSB (BSM!)
- Inflation ($\leq 10^{16}$ GeV)
- Exotic: cosmic strings, primordial black holes, Planck scale
- GR violation: > 2 polarisation states, modified dispersion relation, superluminal propagation, etc.



C. Moore, R. Cole, & C. Berry's GWplotter

Today's question

How can we exploit available GW searches to investigate our newly-developed extra-dimensional model?

What is the significance of these extra dimensions?

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Historically: *A path to unification?*

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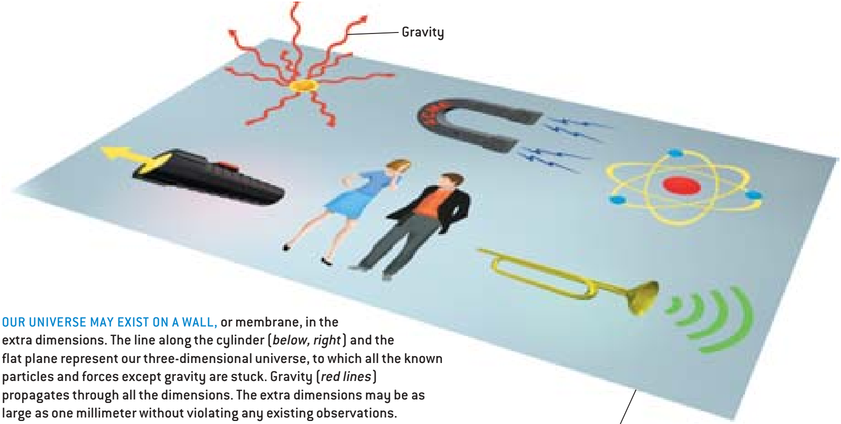
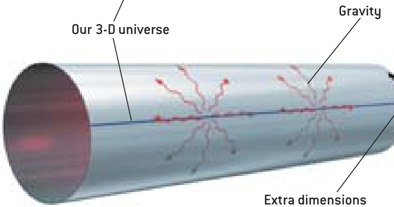
- 1998: Arkani-Hamed, Dimopoulos, Dvali: large EDs



OUR UNIVERSE MAY EXIST ON A WALL, or membrane, in the extra dimensions. The line along the cylinder (*below, right*) and the flat plane represent our three-dimensional universe, to which all the known particles and forces except gravity are stuck. Gravity (*red lines*) propagates through all the dimensions. The extra dimensions may be as large as one millimeter without violating any existing observations.



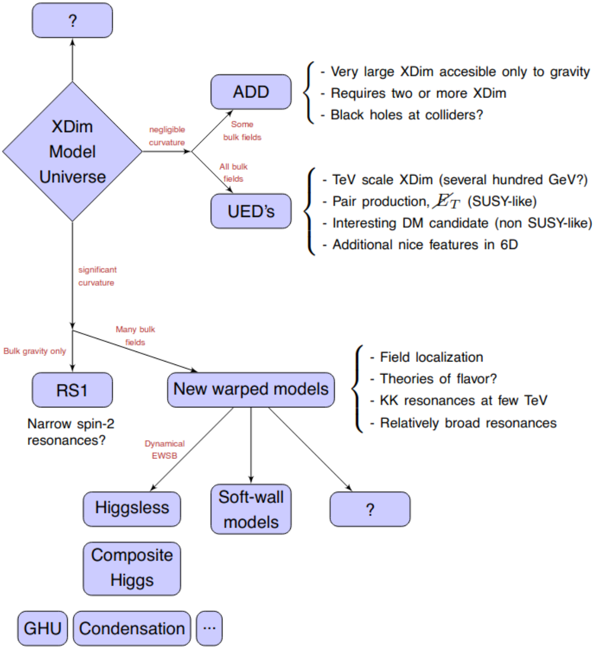
Taken from ADD's 2002 Scientific American article, "The universe's unseen dimensions"

What is the significance of these extra dimensions?

Today: *A path to physics BSM?*

- 1980s: "KK renaissance", 1984 "superstring revolution"
- 1998: Arkani-Hamed, Dimopoulos, Dvali: large EDs
- 1999: Randall & Sundrum: warped EDs

...

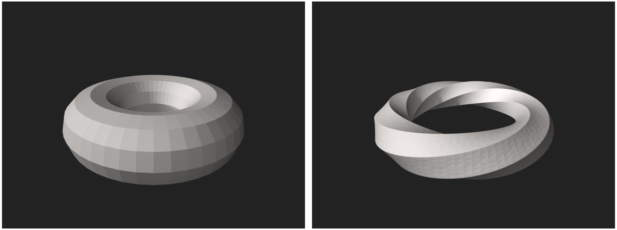


Taken from E. Pontón's 2011 TASI lectures, "TeV scale EDs"

From algebra to phenomenology

How we generate a massive scalar sector from a partially compactified Yang-Mills theory, completely determined by the gauge structure and geometry of a twisted torus

Any Lie group G of dimension d can be understood as a d-dimensional differentiable manifold. To compactify solvable G, we quotient by the lattice Γ. For nilpotent groups, the resultant twisted torus is a nilmanifold.

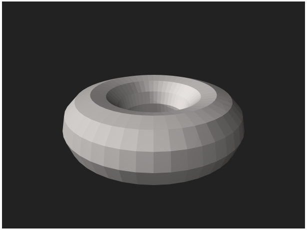
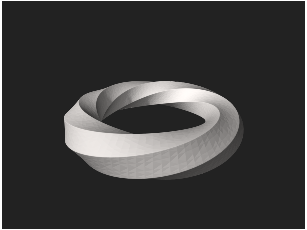


Wikimedia Commons, Torus & Twisted torus

Consider the Lie algebra...

$$[Z_b, Z_c] = f^a{}_{bc}\, Z_a \;, \quad f^a{}_{bc} \; = -f^a{}_{bc}$$

$$\mathcal{R} = -\frac{1}{4}\delta_{ad}\, \delta^{bc}\, \delta^{cg} f^a{}_{bc} f^d{}_{cg}$$

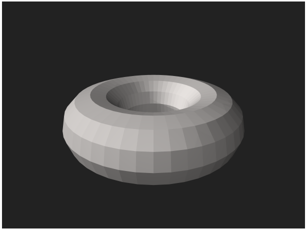
Wikimedia Commons, Torus & Twisted torus

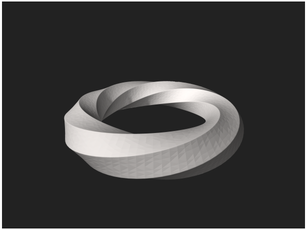
Consider the Heisenberg algebra...

$$[Z_1, Z_2] = -\mathtt{f} Z_3 \;, \quad [Z_1, Z_3] = [Z_2, Z_3] = 0$$

$$\mathrm{d}e^3 = \mathtt{f} e^1 \wedge e^2 \;, \quad \mathrm{d}e^1 = 0 \;, \quad \mathrm{d}e^2 = 0$$

$$e^1 = r^1 dy^1 \;, \quad e^2 = r^2 dy^2 \;, \quad e^3 = r^3 (dy^3 + N r^1 dy^2) \;, \quad N = \frac{r^1 r^2}{r^3} \mathtt{f}$$





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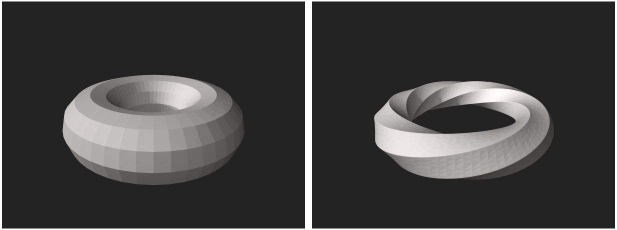
...gives us the metric $ds^2_{\mathcal{N}} = g^{\mathcal{N}}_{ij} dx^i dx^j$



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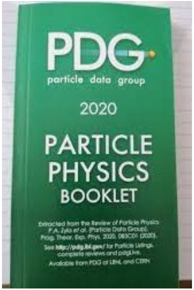
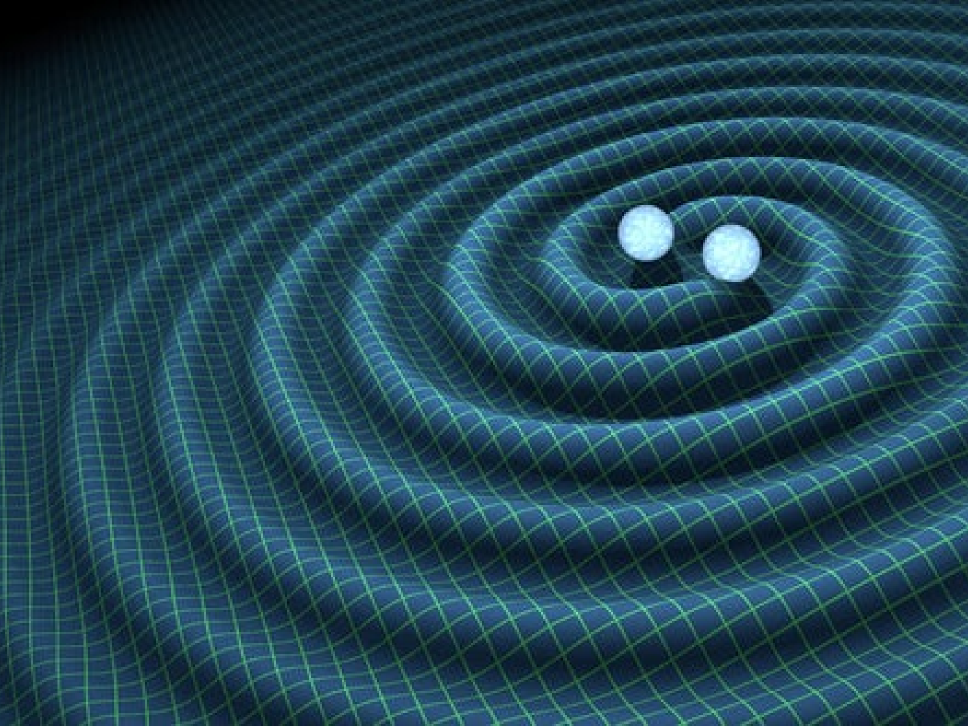
Constraining EDs

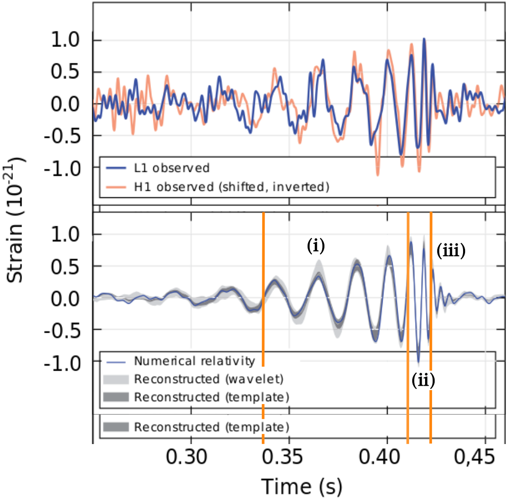
- Limits on R from Deviations in Gravitational Force Law
- Limits on R from On-Shell Production of Gravitons: $\delta = 2$
- Mass Limits on M_{TT}
- Limits on $1/R = M_c$
- Limits on Kaluza-Klein Gravitons in Warped Extra Dimensions
- Limits on Kaluza-Klein Gluons in Warped Extra Dimensions
- Black Hole Production Limits
  – Semiclassical Black Holes
  – Quantum Black Holes

ATLAS, CMS, DELPHI, ALEPH, CDF, D0, OPAL, etc.





Quasinormal mode: "ringdown"



(i) inspiral
(ii) merger
(iii) ringdown

B. P. Abbott *et al.*, PRL **116**, 061102 (2016).

Quasinormal mode and frequency

$$\Psi(x^\mu) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{r} e^{-i\omega t} Y_{\ell m}(\theta,\phi) \ , \qquad \omega_{sn\ell} = \omega_R - in\omega_I$$

- $\mathbb{R}e\{\omega\}$ = physical oscillation frequency
- $\mathbb{I}m\{\omega\}$ = damping $\rightarrow$ dissipative, "quasi"

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- s: spin of perturbing field
- m: azimuthal number for spherical harmonic decomposition in θ_i
- ℓ: angular/multipolar number for spherical harmonic decomposition in θ, ϕ
- n: overtone number labels QNMs by a monotonically increasing $|\mathbb{I}m\{\omega\}|$

Black hole wave equation:

$$\frac{d^2}{dx^2}\varphi(x) + \left[\omega^2 - V(r)\right]\varphi(x) = 0\,, \qquad \frac{dr}{dx} = f(r)$$

$\rightarrow$ reduces to a second-order ODE in r

Black hole wave equation:

$$\frac{d^2}{dx^2}\varphi(x) + \left[\omega^2 - V(r)\right]\varphi(x) = 0\ , \qquad \frac{dr}{dx} = f(r)$$

$\rightarrow$ subjected to **QNM boundary conditions**

purely ingoing:	$\varphi(x) \sim e^{+i\omega x}$	$x \rightarrow -\infty\ (r \rightarrow r_H)$
purely outgoing:	$\varphi(x) \sim e^{-i\omega x}$	$x \rightarrow +\infty\ (r \rightarrow +\infty)$

Waves escape domain of study at the boundaries $\Rightarrow$ dissipative

The 7D metric

$$ds^2_{7D} = g^{BH}_{\mu\nu} dx^\mu dx^\nu + g^{\mathcal{N}}_{ij} dx^i dx^j$$

$$\Psi^s_{n\ell m}(t, r, \theta, \phi, y_1, y_2, y_3) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{e^{i\omega t} r} Y^s_{m\ell}(\theta, \phi) Z(y_1, y_2, y_3)$$

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$$ds^2_{BH} = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(\sin^2 d\theta^2 + d\phi^2)$$

$$f(r) = 1 - 2M/r$$

$$ds^2_{\mathcal{N}} = r_1^2 dy_1^2 + r_2^2 dy_2^2 + r_3^2 (dy_3 + N r_1 dy_2)^2$$

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Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{\mathcal{N}}\right) \sum \Phi(x) Z_k(y) = 0 \ ,$$

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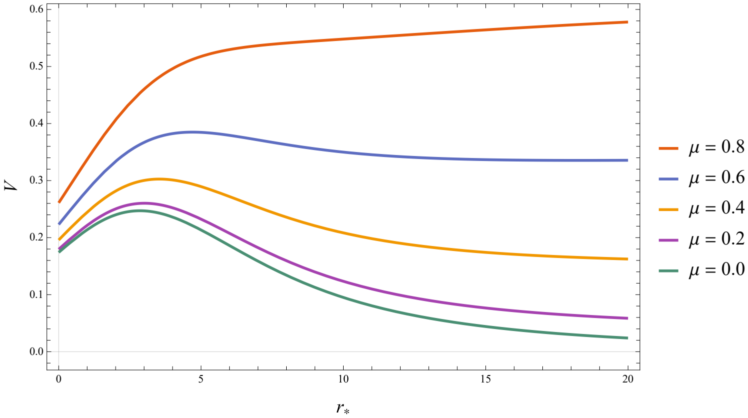
$$\nabla^2 Z_k(y) = -\mu_k^2 Z_k(y)$$

The wavelike equation

$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 - V(r)\right)\psi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right)$$

The fundamental mode: $n = 0$, $\ell = 2$



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μ	ω (WKB)	ω (PT)	ω (DO)
0.0	$0.4836 - 0.0968i$	$0.4874 - 0.0979i$	$0.4836 - 0.0968i$
0.1	$0.4868 - 0.0957i$	$0.4909 - 0.0968i$	$0.4868 - 0.0957i$
0.2	$0.4963 - 0.0924i$	$0.5015 - 0.0936i$	$0.4963 - 0.0924i$
0.3	$0.5123 - 0.0868i$	$0.5192 - 0.0881i$	$0.5124 - 0.0868i$
0.4	$0.5351 - 0.0787i$	$0.5443 - 0.0800i$	$0.5352 - 0.0787i$
0.5	$0.5649 - 0.0676i$	$0.5770 - 0.0690i$	$0.5653 - 0.0676i$
0.6	$0.6022 - 0.0528i$	$0.6181 - 0.0541i$	$0.6032 - 0.0532i$
0.7	$0.1396 + 0.2763i$	$0.6695 - 0.0312i$	$0.6500 - 0.0343i$

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$\Rightarrow$ An upper bound on our QNM probe (“sensitivity range cutoff")

Bounds from GWs?

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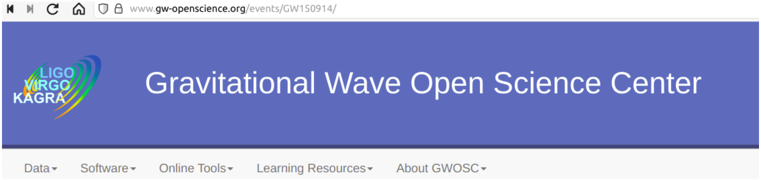
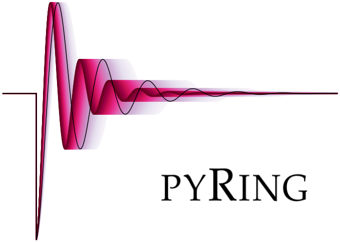


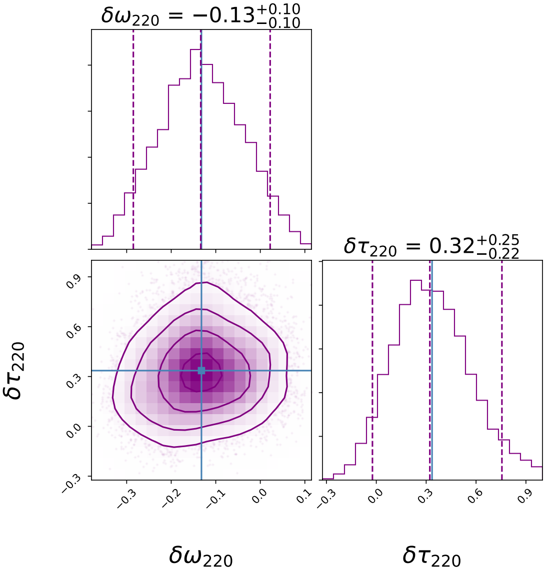


$$\delta\omega = \omega^{GR}(1 + \delta\omega)$$

$$\delta\tau = \tau^{GR}(1 + \delta\tau)$$

The fundamental mode: $n = 0$, $\ell = 2$

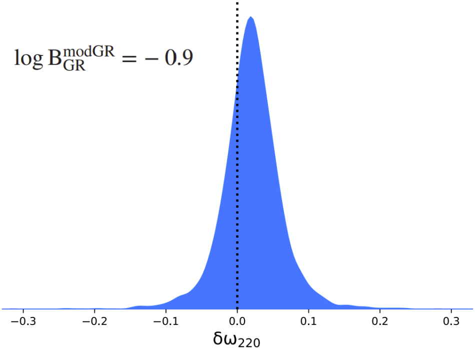
μ	$\delta\omega$	$\delta\tau$
0.0	0.0000	0.0000
0.1	0.0065	0.0113
0.2	0.0262	0.0473
0.3	0.0594	0.1149
0.4	0.1066	0.2302
0.5	0.1687	0.4306
0.6	0.2472	0.8206
0.7	0.3440	1.8181


$\delta\omega_{220} = -0.13^{+0.10}_{-0.10}$
$\delta\tau_{220} = 0.32^{+0.25}_{-0.22}$
$\delta\tau_{220}$
$\delta\omega_{220}$
$\delta\tau_{220}$
0.9
0.6
0.3
0.0
−0.3
−0.3
−0.2
−0.1
0.0
0.1
−0.3
0.0
0.3
0.6
0.9

Most stringent QNM GR deviations

Tests of GR with GWTC-3 [2112.06861]

$$\delta\omega_{O3} = 0.02^{+0.07}_{-0.07}$$
$$\delta\tau_{O3} = 0.13^{+0.21}_{-0.22}$$



$$0.1747 < \mu < 0.3681$$

From the dimensionless parameter $M\mu$,

$$M\mu = \frac{Gm^{\rm BH}m}{\hbar c}$$
$$\Rightarrow \quad m = \frac{1}{m^{\rm BH}}\frac{\hbar c}{G}M\mu$$
$$m \sim 10^{-\chi}10^{-46}{\rm kg} \sim 10^{-(\chi+10)}{\rm eV/c^2}$$

$m \sim 10^{-13}$ eV/c^2 for $M_f \sim 62 M_\odot$ of GW150914

light scalar hypotheses

Conclusions

- Rich phenomenology awaits in the mathematicians′ playground!
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Conclusions

- Rich phenomenology awaits in the mathematicians' playground!
- Connecting theory and observation is non-trivial ("the gap")
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- Using QNM theory, we have introduced a possible new observable + applied naive constraints

  ↪ new model-agnostic search for extra dimensions!

Thank you

Thank you

And a warm thanks to

Backup slides

Why the negativity?

Negatively-curved EDs: a BSM landscape of untapped potential?

Phenomenological implications:

- natural resolution to the hierarchy problem

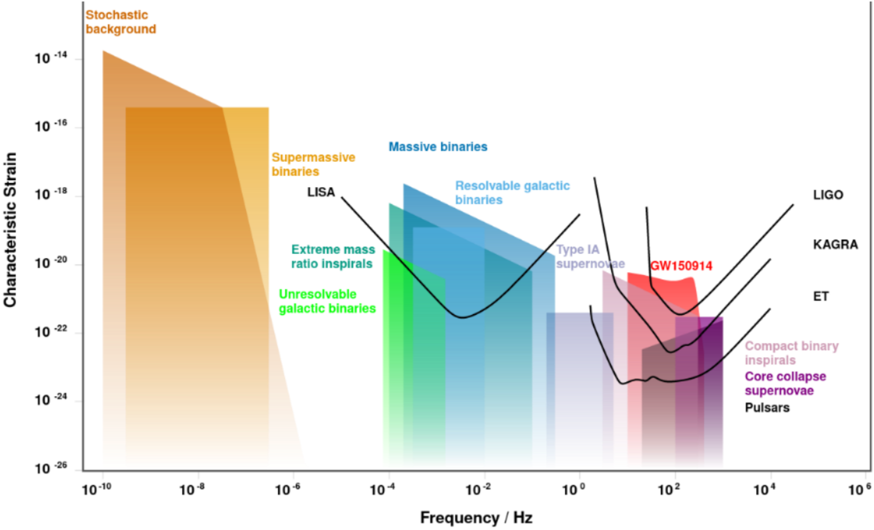
  $\rightarrow$ volume grows exponentially with ℓ_G/ℓ_c

  $\rightarrow$ RSI-like KK mass spectrum w/o light KK modes

- zero modes of Dirac operator emerges w/o gauge breaking
- enables homogeneity & flatness of observed universe

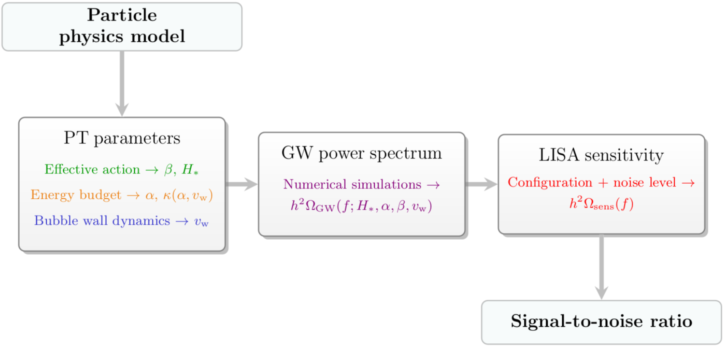
What's next?

Using established techniques to probe the GW BSM landscape...



C. Moore, R. Cole, & C. Berry's GWplotter

Algorithm for finding GWs from PTs



C. Caprini *et al.* JCAP 03 (2020) 024.

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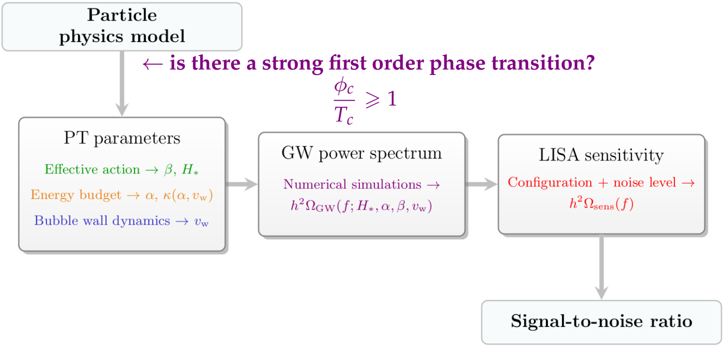
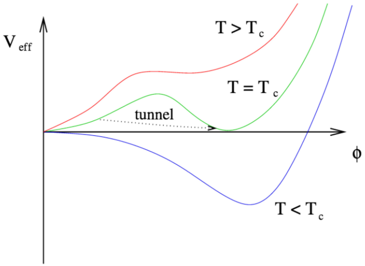
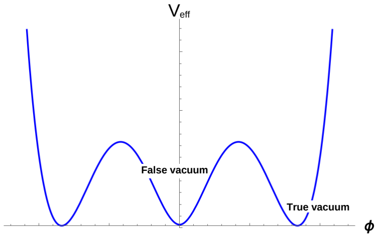


To determine the first order phase transition

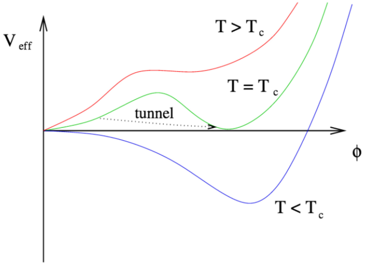


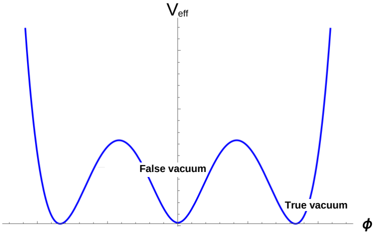


G. White, *A Pedagogical Intro to Baryogenesis*; W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

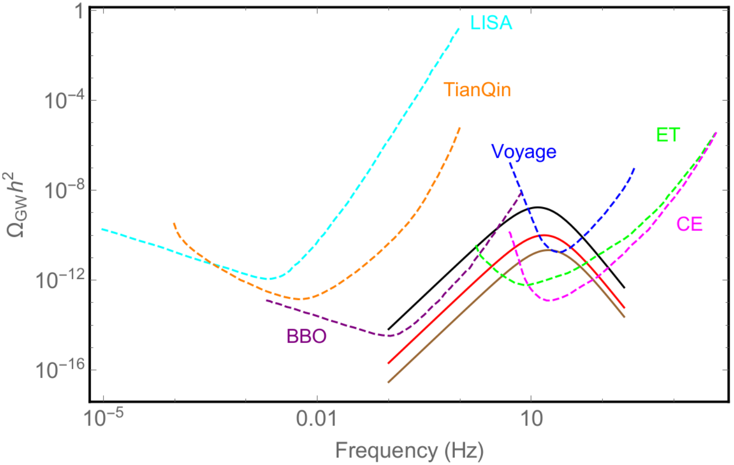
To determine the first order phase transition

$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$



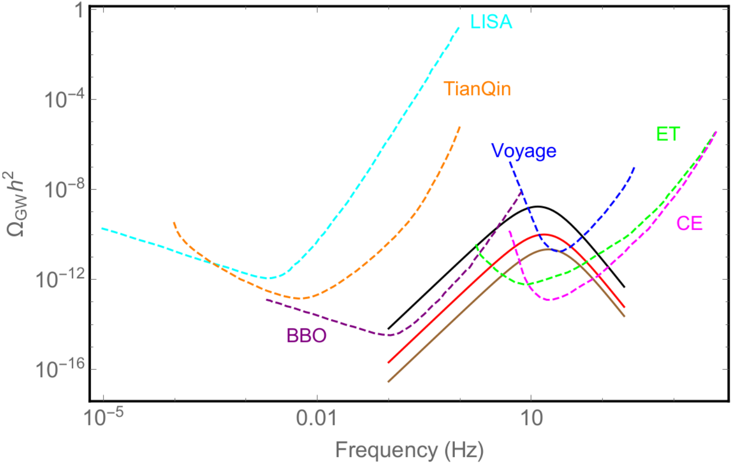


G. White, *A Pedagogical Intro to Baryogenesis*; W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025



W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

Stay tuned!



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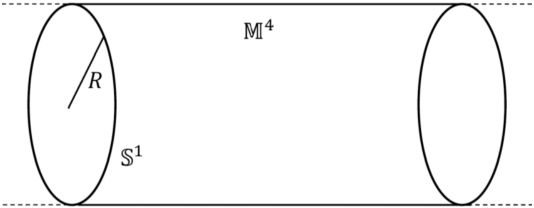
Compactification: infinite $(3+1)$ dims; finite x_5
periodic BCs: $x_5 \to x_5 + 2\pi R$

KK tower of states:

$$\Phi(x^\mu, x^5) = \sum_{n=0}^{\infty} \Phi^{(n)}(x^\mu)\, e^{inx^5/R}\ ,\ m_n = \sqrt{m_0^2 + \left(\frac{n}{R}\right)^2}$$



$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$

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For $\beta = 1/T,\ L_5 = 2\pi R$:

$$\begin{aligned} V_T &= -\frac{3}{4\pi^2}\zeta(5)\frac{1}{L_5^4} \\ &\quad -\frac{3}{4\pi^2}\zeta(5)\frac{L_5}{\beta^2} - \frac{\Gamma(5/2)}{\pi^{5/2}L_5^4}\, 2\sum_{m,n=1}^{\infty}\left[\left(\frac{\beta m}{L_5}\right)^2 + n^2\right]^{-5/2} \end{aligned}$$

$$V_T \sim -\frac{3}{4\pi^2}\zeta(5)\frac{1}{L_5^4} \qquad L_5 \ll \beta$$

$$V_T \sim -\frac{3}{4\pi^2}\zeta(5)\frac{1}{\beta^5} \qquad L_5 \gg \beta$$

Dolan & Ottewill (2009)

A new computation method for BH QNMs through a novel ansatz based on **null geodesics** + expansion of the QNF in inverse powers of $L = \ell + 1/2$

$$\Phi(r) = e^{i\omega z(x)} v(r) \ , \quad \omega = \sum_{k=-1}^{\infty} \omega_k L^{-k}$$

We explore the method for Schwarzschild, RN, and SdS in 4D:

- more efficient means of calculating detectable BH QNMs?
- explore interplay of θ, λ in large-ℓ limit

Components of the ansatz

$$v(r) = \exp\left\{ \sum_{k=0}^{\infty} S_k(r) L^{-k} \right\}, \quad z(x) = \int^x \rho(r) dx = \int^x b_c k_c(r) dx$$

$$k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

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$$r_c = \left. \frac{2f(r)}{\partial_r f(r)} \right|_{r=r_c}, \quad b_c = \left. \sqrt{\frac{r^2}{f(r)}} \right|_{r=r_c}, \quad k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

The DO multipolar expansion method

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We generalise the consequent ODE

$$f(r)\frac{d}{dr}\left(f(r)\frac{dv}{dr}\right) + 2i\omega\rho(r)\frac{dv}{dr} + \left[i\omega f(r)\frac{d\rho}{dr} + \left(1-\rho(r)^2\right)\omega^2 - V(r)\right]v(r) = 0$$

We solve iteratively for ω_k and $S'_k(r)$ and sub into ω_k

QNF expansions for the Schwarzschild BH

$$r_c = 3\ ,\ b_c = \sqrt{27} \quad \Rightarrow \quad \rho(r) = \left(1 - \frac{3}{r}\right)\sqrt{1 + \frac{6}{r}}$$

$$\begin{aligned}
\omega(L,\mu) = &+\frac{1}{3}L - \frac{i}{6}L^0 + \left[\frac{3\mu^2}{2} + \frac{7}{648}\right]L^{-1} \\
&+ \left[\frac{5i\mu^2}{4} - \frac{137i}{23328}\right]L^{-2} + \left[\frac{9\mu^4}{8} - \frac{379\mu^2}{432} + \frac{2615}{3779136}\right]L^{-3} \\
&+ \left[\frac{27i\mu^4}{16} - \frac{2677i\mu^2}{5184} + \frac{590983i}{1088391168}\right]L^{-4} \\
&+ \left[\frac{63\mu^6}{16} - \frac{427\mu^4}{576} + \frac{362587\mu^2}{1259712} - \frac{42573661}{117546246144}\right]L^{-5} \\
&+ \left[\frac{333i\mu^6}{32} + \frac{6563i\mu^4}{6912} + \frac{100404965i\mu^2}{725594112} + \frac{11084613257i}{25389989167104}\right]L^{-6}\ .
\end{aligned}$$

The fundamental mode: $n = 0$, $\ell = 2$

μ	ω (WKB)	ω (PT)	ω (DO)
0.0	$0.4836 - 0.0968i$	$0.4874 - 0.0979i$	$0.4836 - 0.0968i$
0.1	$0.4868 - 0.0957i$	$0.4909 - 0.0968i$	$0.4868 - 0.0957i$
0.2	$0.4963 - 0.0924i$	$0.5015 - 0.0936i$	$0.4963 - 0.0924i$
0.3	$0.5123 - 0.0868i$	$0.5192 - 0.0881i$	$0.5124 - 0.0868i$
0.4	$0.5351 - 0.0787i$	$0.5443 - 0.0800i$	$0.5352 - 0.0787i$
0.5	$0.5649 - 0.0676i$	$0.5770 - 0.0690i$	$0.5653 - 0.0676i$
0.6	$0.6022 - 0.0528i$	$0.6181 - 0.0541i$	$0.6032 - 0.0532i$
0.7	$0.1396 + 0.2763i$	$0.6695 - 0.0312i$	$0.6500 - 0.0343i$

In agreement with massive scalar QNFs of S. Dolan, Phys. Rev. D 76 (2007) 084001

Suppose we place a 4D Schwarzschild black hole within a 7D spacetime, perturbed by a 7D scalar test field of mass μ:

$$\text{KG:}\ \ \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi\right) - \mu^2\Psi = 0\ ,$$

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(x)dx^a dx^b + g_{ij}(y)dx^i dx^j\ ,$$

$$g_{\mu\nu} = \begin{bmatrix} -f(r) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f(r)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2^2 + r_3^2N^2y_1^2 & r_3^2Ny_1 \\ 0 & 0 & 0 & 0 & 0 & r_3^2Ny_1 & r_3^2 \end{bmatrix},$$

where $f(r) = 1 - 2M/r$

Variable-separable QNM solution:

$$\Psi^s_{n\ell m\mu}(t,r,\theta,\phi,y_1,y_2,y_3) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} R^s_{n\ell\mu}(r) Y^s_{m\ell}(\theta,\phi) Z_\mu(y_1,y_2,y_3) e^{i\omega t} \ .$$

Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{nil}\right) \sum \Phi(x) Z_k(y) = 0 \ ,$$

- $\nabla^2 Y^s_{m\ell}(\theta,\phi) = \frac{-\ell(\ell+1)}{r^2} Y^s_{m\ell}(\theta,\phi)$
- $\nabla^2 Z_k(y) = -\mu_k^2 Z_k(y)$

$$\mu^2_{k,j,m} = \frac{4\pi^2 k^2}{(r_3)^2}\left[1 + \frac{(2m+1) r_3}{2\pi |k|} |\mathtt{f}|\right]$$

Table I. Stabilities of generalised static black holes. In this table, "d" represents the spacetime dimension, $n+2$. The results for tensor perturbations apply only for maximally symmetric black holes, while those for vector and scalar perturbations are valid for black holes with generic Einstein horizons, except in the case with $K=1, Q=0, \lambda>0$ and $d=6$.

		Tensor		Vector		Scalar	
		$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$
$K=1$	$\lambda=0$	OK	OK	OK	OK	OK	$d=4,5$ OK $d\geq 6$?
	$\lambda>0$	OK	OK	OK	OK	$d\leq 6$ OK $d\geq 7$?	$d=4,5$ OK $d\geq 6$?
	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=0$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=-1$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?

$$\mathcal{R}_{ED} = (d-3)K\gamma_{ij}$$