The long-wavelength limit of the Boltzmann equation: recent insights in deriving dissipative relativistic fluid dynamics

Dirk H. Rischke

Institut für Theoretische Physik and Frankfurt Institute for Advanced Studies

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

FIAS Frankfurt Institute for Advanced Studies

thanks to:

Barbara Betz, Ioannis Bouras, Gabriel S. Denicol, Carsten Greiner, Pasi Huovinen, Etele Molnár, Harri Niemi, Jorge Noronha, Zhe Xu

Fluid dynamics: degrees of freedom

1. **Net charge current:** $N^\mu = n\, u^\mu + n^\mu$

   u^μ fluid 4-velocity, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

   $g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ (West coast!!) metric tensor

   $n \equiv u^\mu N_\mu$ net charge density in fluid rest frame

   $n^\mu \equiv \Delta^{\mu\nu} N_\nu \equiv N^{<\mu>}$ diffusion current (flow of net charge relative to u^μ), $n^\mu u_\mu = 0$

   $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ, $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:** $T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}$

   $\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ energy density in fluid rest frame

   p pressure in fluid rest frame

   Π bulk viscous pressure, $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$

   $q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ heat flux current (flow of energy relative to u^μ), $q^\mu u_\mu = 0$

   $\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ shear stress tensor, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,,\ \pi^\mu{}_\mu = 0$

   $a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor

   $a^{<\mu\nu>} \equiv \left(\Delta_\alpha{}^{(\mu} \Delta^{\nu)}{}_\beta - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta}\right) a^{\alpha\beta}$ symmetrized, traceless spatial projection

Fluid dynamics: equations of motion

1. **Net charge conservation:**

$$\partial_\mu N^\mu = 0 \quad \Longleftrightarrow \quad \dot{n} + n\,\theta + \partial\cdot n = 0$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative
(time derivative in fluid rest frame, $\dot{a}_{\rm RF} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$\partial_\mu T^{\mu\nu} = 0 \quad \Longleftrightarrow$ energy conservation:

$$u_\nu\,\partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial\cdot q - q\cdot\dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0$$

acceleration equation:

$$\Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \quad \Longleftrightarrow$$

$$(\epsilon+p)\dot{u}^\mu = \nabla^\mu(p+\Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q\cdot\partial u^\mu - \Delta^{\mu\nu}\,\partial^\lambda\pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient,
(spatial gradient in fluid rest frame, $u^\mu_{\rm RF} \equiv (1,0,0,0)$)

Solvability

Problem:

5 equations, but 15 unknowns (for given u^μ): ϵ , p , n , Π , n^μ (3) , q^μ (3) , $\pi^{\mu\nu}$ (5)

Solution:

1. **clever choice of frame (Eckart, Landau,...): eliminate n^μ or q^μ**

   $\Longrightarrow$ **does not help! Promotes u^μ to dynamical variable!**

2. **ideal fluid limit: all dissipative terms vanish, $\Pi = n^\mu = q^\mu = \pi^{\mu\nu} = 0$**

   $\Longrightarrow$ **6 unknowns: ϵ , p , n , u^μ (3) (not quite there yet...)**

   $\Longrightarrow$ **fluid is in local thermodynamical equilibrium**

   $\Longrightarrow$ **provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations**

3. **provide additional equations for dissipative quantities**

   $\Longrightarrow$ **dissipative relativistic fluid dynamics**

   (a) **First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Landau, Lifshitz)**

   (b) **Second-order theories: e.g. Israel-Stewart (IS) equations**

Navier-Stokes equations

Navier-Stokes (NS) equations: first-order, dissipative relativistic fluid dynamics

1. bulk viscous pressure: $\Pi_{\rm NS} = -\zeta\,\theta$

   ζ bulk viscosity

2. diffusion current: $n^\mu_{\rm NS} = \kappa_n\,\nabla^\mu \alpha$

   $\beta \equiv 1/T$ inverse temperature,

   $\alpha \equiv \beta\,\mu$, μ chemical potential,

   κ_n net-charge diffusion coefficient

3. shear stress tensor: $\pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}$

   η shear viscosity,

   $\sigma^{\mu\nu} = \nabla^{<\mu} u^{\nu>}$ shear tensor

$\Longrightarrow$ algebraic expressions in terms of thermodynamic and fluid variables

$\Longrightarrow$ simple... but: unstable and acausal equations of motion!!

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725

Israel-Stewart equations

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" version:

$$\tau_\Pi \, \dot{\Pi} + \Pi = \Pi_{\rm NS}$$
$$\tau_n \, \dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS}$$
$$\tau_\pi \, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS}$$

$\Longrightarrow$ **dynamical (instead of algebraic) equations for dissipative terms!**

solution: e.g. bulk viscous pressure $\Pi(t) = \Pi_{\rm NS} \left(1 - e^{-t/\tau_\Pi}\right) + \Pi(0)\, e^{-t/\tau_\Pi}$

$\Longrightarrow$ **dissipative quantities** $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$ **relax to their respective NS values** $\Pi_{\rm NS}\,,\ n^\mu_{\rm NS}\,,\ \pi^{\mu\nu}_{\rm NS}$ **on time scales** $\tau_\Pi\,,\ \tau_n\,,\ \tau_\pi$

$\Longrightarrow$ **stable and causal fluid dynamical equations of motion!**

see, e.g., S. Pu, T. Koide, DHR, PRD81 (2010) 114039

Power counting (I)

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength $\lambda_{\mathrm{th}} \sim \beta \equiv 1/T$
- mean free path $\ell_{\mathrm{mfp}} \sim (\langle\sigma\rangle n)^{-1}$

  $\langle\sigma\rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\mathrm{th}}^{-3}$
- length scale over which macroscopic fluid fields vary L_{hydro} , $\partial_\mu \sim L_{\mathrm{hydro}}^{-1}$

Note: since $\eta \sim (\langle\sigma\rangle\lambda_{\mathrm{th}})^{-1}$ $\Longrightarrow$
$$\boxed{\frac{\ell_{\mathrm{mfp}}}{\lambda_{\mathrm{th}}} \sim \frac{1}{\langle\sigma\rangle n}\,\frac{1}{\lambda_{\mathrm{th}}} \sim \frac{\lambda_{\mathrm{th}}^3}{\langle\sigma\rangle\lambda_{\mathrm{th}}} \sim \frac{\lambda_{\mathrm{th}}^3}{\langle\sigma\rangle\lambda_{\mathrm{th}}} \sim \frac{\eta}{s}}$$

s entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\mathrm{th}}^{-3}$

$\Longrightarrow$ $\dfrac{\eta}{s}$ solely determined by the 2 microscopic length scales!

Note: similar argument holds for $\dfrac{\zeta}{s}$, $\dfrac{\kappa_n}{\beta\, s}$

Power counting (II)

3 regimes:

- dilute gas limit $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{\eta}{s} \gg 1 \iff \langle\sigma\rangle \ll \lambda_{\rm th}^2 \implies$ weak-coupling limit
- viscous fluids $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{\eta}{s} \sim 1 \iff \langle\sigma\rangle \sim \lambda_{\rm th}^2$

  interactions happen on the scale $\lambda_{\rm th}$ $\implies$ moderate coupling
- ideal fluid limit $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{\eta}{s} \ll 1 \iff \langle\sigma\rangle \gg \lambda_{\rm th}^2 \implies$ strong-coupling limit

gradient (derivative) expansion: $$\boxed{\ell_{\rm mfp}\,\partial_\mu \sim \frac{\ell_{\rm mfp}}{L_{\rm hydro}} \equiv K \sim \delta \ll 1}$$

K Knudsen number

$\implies$ equivalent to $k\,\ell_{\rm mfp} \ll 1\,,\quad k$ typical momentum scale

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

$\implies$ separation of macroscopic fluid dynamics (large scale $\sim L_{\rm hydro}$)
from microscopic particle dynamics (small scale $\sim \ell_{\rm mfp}$)

Power counting (III)

Primary quantities: $\epsilon\,,\ p\,,\ n\,,\ s$ $\Longleftrightarrow$ **Dissipative quantities:** $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$

If $K \sim \ell_{\rm mfp}\,\partial_\mu \sim \delta \ll 1$ **, then** $\dfrac{\Pi}{\epsilon} \sim \dfrac{n^\mu}{s} \sim \dfrac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$

Dissipative quantities are small compared to primary quantities

$\Longrightarrow$ **small deviations from local thermodynamical equilibrium!**

Note: statement independent of value of $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa_n}{\beta\, s}\,,\ \dfrac{\eta}{s}$ **!**

Proof: **Gibbs relation:** $\epsilon + p = Ts + \mu n \quad \Longrightarrow \quad \beta\,\epsilon \sim s$ **!**

Estimate dissipative terms by their Navier-Stokes values:

$\Pi \sim \Pi_{\rm NS} = -\zeta\,\theta\,,\quad n^\mu \sim n^\mu_{\rm NS} = \kappa_n\,\nabla^\mu\alpha\,,\quad \pi^{\mu\nu} \sim \pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}$

$$\Longrightarrow \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\,\epsilon}\,\beta\,\theta \sim -\frac{\zeta}{s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\theta \sim \ell_{\rm mfp}\,\partial_\mu u^\mu \sim \delta\,,$$

$$\frac{n^\mu}{s} \sim \frac{\kappa_n}{s}\,\nabla^\mu\alpha \sim \frac{\kappa_n}{\beta\,s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\nabla^\mu\alpha \sim \ell_{\rm mfp}\,\nabla^\mu\alpha \sim \delta\,,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2\,\frac{\eta}{\beta\,\epsilon}\,\beta\,\sigma^{\mu\nu} \sim 2\,\frac{\eta}{s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\sigma^{\mu\nu} \sim \ell_{\rm mfp}\,\nabla^{<\mu}u^{\nu>} \sim \delta\,,\qquad \text{q.e.d.}$$

Israel-Stewart equations revisited (I)

additional relaxation term in IS equation is of second order in δ:

$$\frac{1}{\epsilon}\,\tau_\Pi\,\dot{\Pi} \sim \frac{1}{\epsilon}\,u^\mu\,\ell_{\rm mfp}\,\partial_\mu \Pi \sim K\,\frac{\Pi}{\epsilon} \sim K\,\delta \sim O(\delta^2)$$

$\Longrightarrow$ to be consistent, have to include other second-order terms as well!

$$\begin{aligned}
\tau_\Pi\,\dot{\Pi} + \Pi &= \Pi_{\rm NS} + \mathcal{K} \\
\tau_n\,\dot{n}^{<\mu>} + n^\mu &= n^\mu_{\rm NS} + \mathcal{K}^\mu \\
\tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu}
\end{aligned}$$

$$\mathcal{K} = \zeta_1\,\omega_{\mu\nu}\,\omega^{\mu\nu} + \zeta_2\,\sigma^{\mu\nu}\,\sigma_{\mu\nu} + \zeta_3\,\theta^2 + \zeta_4\,(\nabla\alpha_0)^2 + \zeta_5\,(\nabla p)^2 + \zeta_6\,\nabla\alpha_0\cdot\nabla p + \zeta_7\,\nabla^2\alpha_0 + \zeta_8\,\nabla^2 p\,,$$

$$\mathcal{K}^\mu = \kappa_1\,\sigma^{\mu\nu}\,\nabla_\nu\alpha_0 + \kappa_2\,\sigma^{\mu\nu}\,\nabla_\nu p + \kappa_3\,\theta\,\nabla^\mu\alpha_0 + \kappa_4\,\theta\,\nabla^\mu p + \kappa_5\,\omega^{\mu\nu}\,\nabla_\nu\alpha_0 + \kappa_6\,\Delta^{\mu\lambda}\partial^\nu\sigma_{\lambda\nu} + \kappa_7\,\nabla^\mu\theta\,,$$

$$\begin{aligned}
\mathcal{K}^{\mu\nu} &= \eta_1\,\omega_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} + \eta_2\,\theta\,\sigma^{\mu\nu} + \eta_3\,\sigma_\lambda{}^{<\mu}\,\sigma^{\nu>\lambda} + \eta_4\,\sigma_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} + \eta_5\,\nabla^{<\mu}\alpha\,\nabla^{\nu>}\alpha_0 \\
&\quad + \eta_6\,\nabla^{<\mu}p\,\nabla^{\nu>}p + \eta_7\,\nabla^{<\mu}\alpha\,\nabla^{\nu>}p + \eta_8\,\nabla^{<\mu}\nabla^{\nu>}\alpha_0 + \eta_9\,\nabla^{<\mu}\nabla^{\nu>}p
\end{aligned}$$

where $\omega^{\mu\nu} \equiv \nabla^{<\mu}u^{\nu>}$ vorticity

$\Longrightarrow$ second-order gradient expansion!

cf. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100
P. Romatschke, Class. Quant. Grav. 27 (2010) 025006

Israel-Stewart equations revisited (II)

unfortunately, including second-order gradient terms renders eqs. of motion parabolic

$\Longrightarrow$ acausal, unstable $\Longrightarrow$ in general, $\mathcal{K}$, $\mathcal{K}^\mu$, $\mathcal{K}^{\mu\nu}$ have to be omitted!

. . . but there is more: in principle, Π, n^μ, $\pi^{\mu\nu}$ are quantities independent from

$$\theta\,,\ \nabla^\mu \alpha_0\,,\ \nabla^\mu p\,,\ \sigma^{\mu\nu}\,,\ \omega^{\mu\nu}$$

$\Longrightarrow$ additional Lorentz-covariants:

$$\tau_\Pi\,\dot{\Pi} + \Pi = \Pi_{\rm NS} + \mathcal{K} + \mathcal{J} + \mathcal{R}$$

$$\tau_n\,\dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS} + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu$$

$$\tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$$

$$\mathcal{J} = -\ell_{\Pi n}\,\nabla\cdot n - \tau_{\Pi n}\,n\cdot\nabla p - \delta_{\Pi\Pi}\,\theta\,\Pi - \lambda_{\Pi n}\,n\cdot\nabla\alpha_0 + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu}$$

$$\mathcal{J}^\mu = \omega^{\mu\nu}\,n_\nu - \delta_{nn}\,\theta\,n^\mu - \ell_{n\Pi}\,\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\,\nabla^\lambda\pi_{\nu\lambda} + \tau_{n\Pi}\,\Pi\,\nabla^\mu p - \tau_{n\pi}\,\pi^{\mu\nu}\,\nabla_\nu p - \lambda_{nn}\,\sigma^{\mu\nu}\,n_\nu + \lambda_{n\Pi}\,\Pi\,\nabla^\mu\alpha_0 - \lambda_{n\pi}\,\pi^{\mu\nu}\,\nabla_\nu\alpha_0$$

$$\mathcal{J}^{\mu\nu} = 2\,\pi_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} - \delta_{\pi\pi}\,\theta\,\pi^{\mu\nu} - \tau_{\pi\pi}\,\pi_\lambda{}^{<\mu}\,\sigma^{\nu>\lambda} + \lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu} - \tau_{\pi n}\,n^{<\mu}\,\nabla^{\nu>}p + \ell_{\pi n}\,\nabla^{<\mu}n^{\nu>} + \lambda_{\pi n}\,n^{<\mu}\nabla^{\nu>}\alpha_0$$

$$\mathcal{R} = \varphi_1\,\Pi^2 + \varphi_2\,n\cdot n + \varphi_3\,\pi^{\mu\nu}\pi_{\mu\nu}$$

$$\mathcal{R}^\mu = \varphi_4\,\pi^{\mu\nu}\,n_\nu + \varphi_5\,\Pi\,n^\mu$$

$$\mathcal{R}^{\mu\nu} = \varphi_6\,\Pi\,\pi^{\mu\nu} + \varphi_7\,\pi_\lambda{}^{<\mu}\,\pi^{\nu>\lambda} + \varphi_8\,n^{<\mu}\,n^{\nu>}$$

Matching to kinetic theory

Fluid dynamics is an effective (macroscopic) theory for the long-wavelength, small-frequency limit of a given (microscopic) theory

$\Longrightarrow$ **coefficients in equations of motion can be determined by matching to the underlying theory, e.g. kinetic theory**

G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD85 (2012) 114047

Details (I)

1. Boltzmann equation $k \cdot \partial f_k = C[f]$ for single-particle distribution function f_k
2. introduce irreducible tensors of rank ℓ: $k_{<\mu_1} \cdots k_{\mu_\ell>} \equiv \Delta^{\nu_1 \cdots \nu_\ell}_{\mu_1 \cdots \mu_\ell} k_{\nu_1} \cdots k_{\nu_\ell}$

   $\Delta^{\nu_1 \cdots \nu_\ell}_{\mu_1 \cdots \mu_\ell}$ are projectors onto subspaces orthogonal to u^μ, symmetric in μ_i, ν_i, and traceless
3. f_k can be expanded in terms of $k_{<\mu_1} \cdots k_{\mu_\ell>}$

$$f_k = f_{0k} + f_{0k}\, \tilde{f}_{0k} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}^{(\ell)}_{kn}\, \rho^{\mu_1 \cdots \mu_\ell}_n\, k_{<\mu_1} \cdots k_{\mu_\ell>}$$

where

(a) $f_{0k} = [\exp(\beta\, u \cdot k - \alpha) + a]^{-1}$ single-particle distribution function in local equilibrium, $a = \pm 1/0$ for Fermi/Bose/Boltzmann statistics

(b) $\tilde{f}_{0k} = 1 - a\, f_{0k}$

(c) $\mathcal{H}^{(\ell)}_{kn} = \dfrac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a^{(\ell)}_{mn}\, P^{(\ell)}_{km}$, where

$P^{(\ell)}_{kn} = \sum_{r=0}^{n} a^{(\ell)}_{nr}\, E^r_k$ are orthogonal polynomials of order n in energy $E_k \equiv u \cdot k$

$\Longrightarrow$ $\mathcal{H}^{(\ell)}_{kn}$ are polynomials of order N_ℓ in energy E_k

(d) irreducible moments of $\delta f_k \equiv f_k - f_{0k}$: $\rho^{\mu_1 \cdots \mu_\ell}_n = \int \mathrm{d}K\, \delta f_k\, E^n_k\, k^{<\mu_1} \cdots k^{\mu_\ell>}$

Details (II)

4. rewrite Boltzmann equation in the form $\delta \dot{f}_k = -\dot{f}_{0k} - \frac{1}{E_k} \{ k \cdot \nabla (f_{0k} + \delta f_k) - C[f] \}$

5. derive equations of motion for irreducible moments, e.g. up to $\ell = 2$:

$$\begin{aligned}
\dot{\rho}_r &= C_{r-1} + \alpha_r^{(0)}\,\theta - \frac{G_{2r}}{D_{20}}\,\theta\,\Pi + \frac{G_{2r}}{D_{20}}\,\sigma^{\mu\nu}\,\pi_{\mu\nu} + \frac{G_{3r}}{D_{20}}\,\partial\cdot n + (r-1)\sigma_{\mu\nu}\,\rho_{r-2}^{\mu\nu} + r\dot{u}_\mu \rho_{r-1}^{\mu} \\
&\quad - \nabla_\mu \rho_{r-1}^{\mu} - \frac{1}{3}\left[(r+2)\rho_r - (r-1)m^2\rho_{r-2}\right]\theta \\
\dot{\rho}_r^{<\mu>} &= C_{r-1}^{<\mu>} + \alpha_r^{(1)}\,\nabla^\mu\alpha + \omega^\mu{}_\nu \rho_r^\nu - \frac{1}{3}\left[(r+3)\rho_r^\mu - (r-1)m^2\rho_{r-2}^\mu\right]\theta - \Delta_\lambda^\mu \nabla_\nu \rho_{r-1}^{\lambda\nu} \\
&\quad - \frac{1}{5}\left[(2r+3)\rho_r^\nu - 2(r-1)m^2\rho_{r-2}^\nu\right]\sigma_\nu^\mu - \frac{1}{3}\left[(r+3)\rho_{r+1} - rm^2\rho_{r-1}\right]\dot{u}^\mu \\
&\quad + \frac{\beta J_{r+2,1}}{\epsilon+p}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta^{\mu\nu}\partial^\lambda\pi_{\lambda\nu}\right) + \frac{1}{3}\nabla^\mu\left(\rho_{r+1} - m^2\rho_{r-1}\right) \\
&\quad + (r-1)\rho_{r-2}^{\mu\nu\lambda}\sigma_{\lambda\nu} + r\dot{u}_\nu\rho_{r-1}^{\mu\nu} \\
\dot{\rho}_r^{<\mu\nu>} &= C_{r-1}^{<\mu\nu>} + 2\,\alpha_r^{(2)}\sigma^{\mu\nu} - \frac{2}{7}\left[(2r+5)\rho_r^{\lambda<\mu} - 2(r-1)m^2\rho_{r-2}^{\lambda<\mu}\right]\sigma_\lambda^{\nu>} + 2\rho_r^{\lambda<\mu}\omega_\lambda{}^{\nu>} \\
&\quad + \frac{2}{15}\left[(r+4)\rho_{r+2} - (2r+3)m^2\rho_r + (r-1)m^4\rho_{r-2}\right]\sigma^{\mu\nu} \\
&\quad + \frac{2}{5}\nabla^{<\mu}\left(\rho_{r+1}^{\nu>} - m^2\rho_{r-2}^{\nu>}\right) - \frac{2}{5}\left[(r+5)\rho_{r+1}^{<\mu} - rm^2\rho_{r-1}^{<\mu}\right]\dot{u}^{\nu>} \\
&\quad - \frac{1}{3}\left[(r+4)\rho_r^{\mu\nu} - (r-1)m^2\rho_{r-2}^{\mu\nu}\right]\theta + (r-1)\rho_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda
\end{aligned}$$

$\alpha_r^{(\ell)}$, G_{nm}, D_{nq}, J_{nq} thermodynamic functions

$C_r^{<\mu_1\cdots\mu_\ell>} = \int dK\, E_k^r\, k^{<\mu_1}\cdots k^{\mu_\ell>}\, C[f]$ irreducible moment of collision integral

Details (III)

Remarks:

(a) system of infinitely many coupled equations for irreducible moments $\rho_r^{\mu_1 \cdots \mu_\ell}$

(b) system completely equivalent to Boltzmann equation

(c) by definition $\rho_0 = -\frac{3}{m^2}\Pi\,,\ \rho_0^\mu = n^\mu\,,\ \rho_0^{\mu\nu} = \pi^{\mu\nu}$

(d) matching conditions in Landau frame imply $\rho_1 = \rho_2 = \rho_1^\mu = 0$

6. fluid dynamics comprises tensors up to rank 2 $\Longrightarrow$ neglect $\rho_r^{\mu_1 \cdots \mu_\ell}$ with $\ell > 2$

7. linearize collision integral: $C_{r-1}^{<\mu_1 \cdots \mu_\ell>} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)}\, \rho_n^{\mu_1 \cdots \mu_\ell} + O(\delta f_k^2)$

$\Longrightarrow$ linearized equation of motion for irreducible moments:

$$\dot{\vec{\rho}} + \mathcal{A}^{(0)}\,\vec{\rho} \simeq \vec{\alpha}^{(0)}\theta + \ldots$$
$$\dot{\vec{\rho}}^{\,\mu} + \mathcal{A}^{(1)}\,\vec{\rho}^{\,\mu} \simeq \vec{\alpha}^{(1)}\nabla^\mu\alpha + \ldots$$
$$\dot{\vec{\rho}}^{\,\mu\nu} + \mathcal{A}^{(2)}\,\vec{\rho}^{\,\mu\nu} \simeq 2\,\vec{\alpha}^{(2)}\sigma^{\mu\nu} + \ldots$$

8. diagonalize collision matrix: $(\Omega^{-1})^{(\ell)}\mathcal{A}^{(\ell)}\Omega^{(\ell)} = \mathrm{diag}(\chi_0^{(\ell)}, \ldots, \chi_j^{(\ell)}, \ldots)$

for later purposes: $\tau^{(\ell)} \equiv \left(\mathcal{A}^{-1}\right)^{(\ell)} = \Omega^{(\ell)}\ \mathrm{diag}\left(1/\chi_0^{(\ell)}\,,\ \ldots\,,\ 1/\chi_j^{(\ell)}\,,\ \ldots\right)\ \left(\Omega^{-1}\right)^{(\ell)}$

$\Longrightarrow$

$$\tau^{(0)}\,\dot{\vec{\rho}} + \vec{\rho} \simeq \tau^{(0)}\,\vec{\alpha}^{(0)}\theta + \ldots$$
$$\tau^{(1)}\,\dot{\vec{\rho}}^{\,\mu} + \vec{\rho}^{\,\mu} \simeq \tau^{(1)}\,\vec{\alpha}^{(1)}\nabla^\mu\alpha + \ldots$$
$$\tau^{(2)}\,\dot{\vec{\rho}}^{\,\mu\nu} + \vec{\rho}^{\,\mu\nu} \simeq 2\,\tau^{(2)}\,\vec{\alpha}^{(2)}\sigma^{\mu\nu} + \ldots$$

Details (IV)

9. eigenmodes of linearized equations of motion: $X_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} (\Omega^{-1})_{ij}^{(\ell)} \rho_j^{\mu_1\cdots\mu_\ell}$

$\Longrightarrow$ equations of motion for eigenmodes decouple:

$$\dot{X}_i + \chi_i^{(0)} X_i = \beta_i^{(0)} \theta + \ldots$$
$$\dot{X}_i^{<\mu>} + \chi_i^{(1)} X_i^\mu = \beta_i^{(1)} \nabla^\mu \alpha + \ldots$$
$$\dot{X}_i^{<\mu\nu>} + \chi_i^{(2)} X_i^{\mu\nu} = \beta_i^{(2)} \sigma^{\mu\nu} + \ldots$$

where $\beta_i^{(0)} = \sum_{j=0,\neq 1,2}^{N_0} (\Omega^{-1})_{ij}^{(0)} \alpha_j^{(0)}$, $\beta_i^{(1)} = \sum_{j=0,\neq 1}^{N_1} (\Omega^{-1})_{ij}^{(1)} \alpha_j^{(1)}$, $\beta_i^{(2)} = 2 \sum_{j=0}^{N_2} (\Omega^{-1})_{ij}^{(2)} \alpha_j^{(2)}$

10. slowest eigenmodes (w/o r.o.g. $i = 0$) remain dynamical,
all faster ones ($i \neq 0$) are replaced by their asymptotic (NS) values:

$$X_i \simeq \frac{\beta_i^{(0)}}{\chi_i^{(0)}} \theta \; , \quad X_i^\mu \simeq \frac{\beta_i^{(1)}}{\chi_i^{(1)}} \nabla^\mu \alpha \; , \quad X_i^{\mu\nu} \simeq \frac{\beta_i^{(2)}}{\chi_i^{(2)}} \sigma^{\mu\nu}$$

11. Since $\rho_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell}$:

$$\rho_i \simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta$$
$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu \alpha$$
$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

Details (V)

$\Longrightarrow$ express X_0 , X_0^μ , $X_0^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu \alpha$, $\sigma^{\mu\nu}$

(w/o r.o.g. $\Omega_{00}^{(\ell)} \equiv 1$):

$$X_0 \simeq -\frac{3}{m^2}\Pi - \sum_{j=3}^{N_0} \Omega_{0j}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}}\theta$$

$$X_0^\mu \simeq n^\mu - \sum_{j=2}^{N_1} \Omega_{0j}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu \alpha$$

$$X_0^{\mu\nu} \simeq \pi^{\mu\nu} - \sum_{j=1}^{N_2} \Omega_{0j}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

$\Longrightarrow$ express ρ_i , ρ_i^μ , $\rho_i^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$:

$$\frac{m^2}{3}\rho_i \simeq -\Omega_{i0}^{(0)}\Pi + \left(\zeta_i - \Omega_{i0}^{(0)}\zeta_0\right)\theta$$

$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_{n\,i} - \Omega_{i0}^{(1)}\kappa_{n\,0}\right)\nabla^\mu\alpha$$

$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)}\pi^{\mu\nu} + 2\left(\eta_i - \Omega_{i0}^{(2)}\eta_0\right)\sigma^{\mu\nu}$$

where $\zeta_i = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{ir}^{(0)} \alpha_r^{(0)}$, $\kappa_{n\,i} = \sum_{r=0,\neq 1}^{N_1} \tau_{ir}^{(1)}\alpha_r^{(1)}$, $\eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)}\alpha_r^{(2)}$

$\Longrightarrow$ equations of motion for irreducible moments become identical with equations of motion for dissipative quantities Π , n^μ , $\pi^{\mu\nu}$

$\Longrightarrow$ identify transport coefficients

Discussion (I)

1. Basis of expansion for δf_k is orthogonal in irreducible subspaces
   $\Longrightarrow$ truncation at any order in ℓ and N_ℓ possible!
2. 14-moment approximation corresponds to choice $N_0 = 2\,,\ N_1 = 1\,,\ N_2 = 0$ and leads to IS equations
3. approximation can be systematically improved by increasing N_ℓ
4. transport coefficients approach Chapman-Enskog values already for
   $N_0 = 5\,,\ N_1 = 4\,,\ N_2 = 3$ (41-moment approximation)
   Example: classical massless gas with constant cross section σ, $\ell_{\text{mfp}} = (\sigma n)^{-1}$

# of moments	η	$\tau_\pi[\ell_{\text{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
14	$4/(3\sigma\beta)$	$5/3$	$10/7$	0	$4/3$	0	0
23	$14/(11\sigma\beta)$	2	$134/77$	$0.344/\beta$	$4/3$	$-0.689/\beta$	$-0.689/n$
32	$1.268/(\sigma\beta)$	2	1.69	$0.254/\beta$	$4/3$	$-0.687/\beta$	$-0.687/n$
41	$1.267/(\sigma\beta)$	2	1.69	$0.244/\beta$	$4/3$	$-0.685/\beta$	$-0.685/n$

# of moments	κ_n	$\tau_n[\ell_{\text{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/(16\sigma)$	$9/4$	1	$3/5$	$\beta/20$	$\beta/20$	0
23	$21/(128\sigma)$	2.59	1.0	0.96	0.054β	0.118β	$0.0295\beta/p$
32	$0.1605/\sigma$	2.57	1.0	0.93	0.052β	0.119β	$0.0297\beta/p$
41	$0.1596/\sigma$	2.57	1.0	0.92	0.052β	0.119β	$0.0297\beta/p$

Discussion (II)

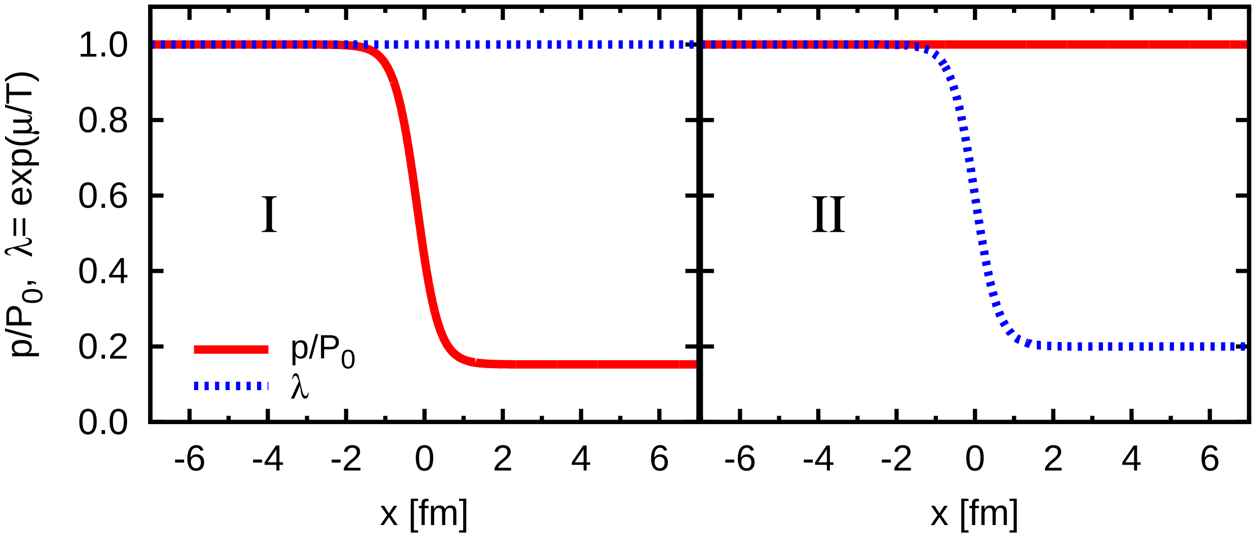
5. approach can be further systematically improved:
   (a) consider also faster eigenmodes X_i, X_i^{μ}, $X_i^{\mu\nu}$, $i > 0$, to be dynamical
   (b) take into account irreducible moments of tensor rank $\ell > 2$
   (c) take into account second-order corrections in the collision integral (compute coefficients $\varphi_1, \ldots, \varphi_8$)

Application: heat-flow problem (I)

G.S. Denicol, H. Niemi, I. Bouras, E. Molnár, Z. Xu, DHR, C. Greiner, arXiv:1207.6811[nucl-th]

Initial conditions: discontinuity in

(I) pressure **(II) fugacity**



$\Longrightarrow$ first-order (NS) terms can be vanishingly small:

$$\text{(I): } \nabla^\mu \alpha \simeq 0 \qquad \text{(II): } \nabla^\mu p \simeq \dot{u}^\mu \simeq 0 \implies \sigma^{\mu\nu} \simeq 0$$

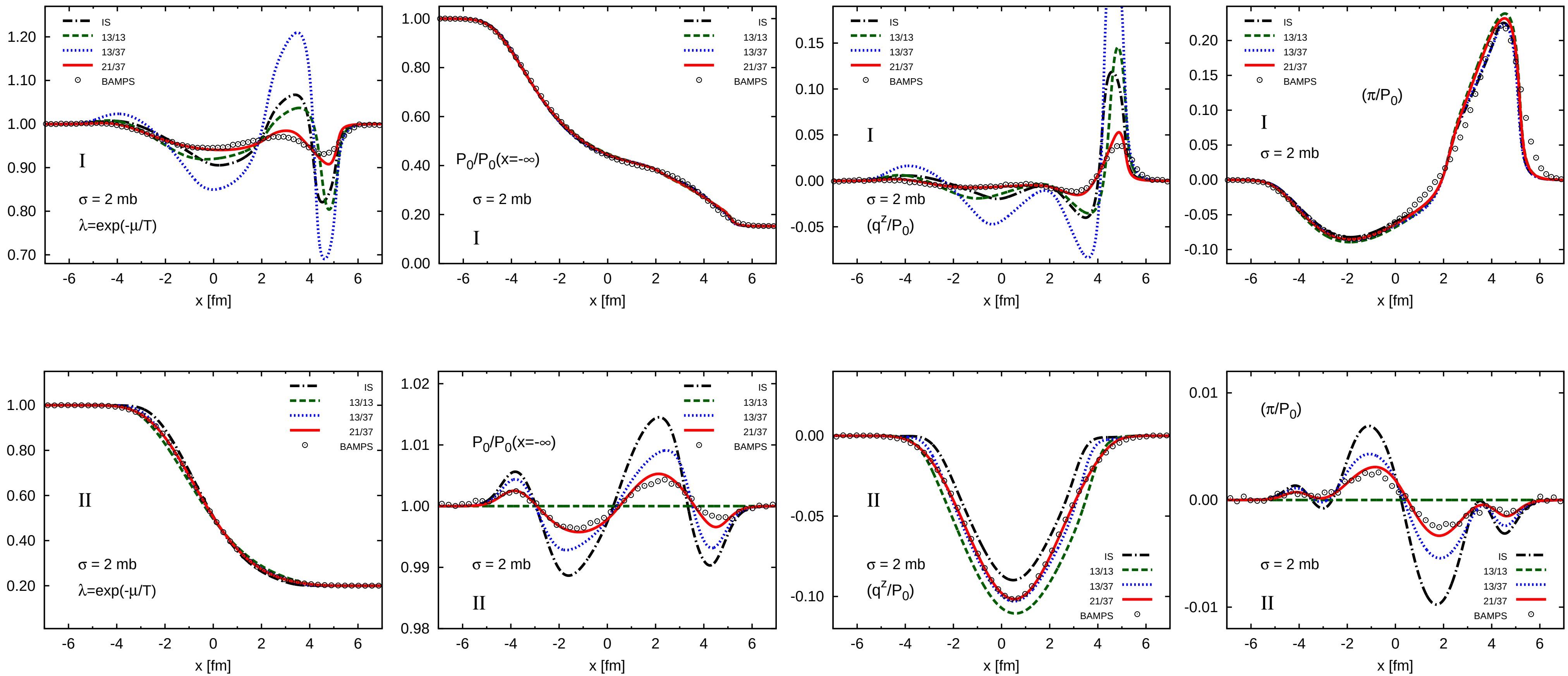
$\Longrightarrow$ second-order terms can become larger than first-order terms!

$\Longrightarrow$ power-counting scheme in terms of Knudsen number is invalidated!

Application: heat-flow problem (II)

Solution: consider ρ_2^{μ} , $\rho_1^{\mu\nu}$ as additional dynamical variables!

$e^{-\alpha}$ p q^{μ} $\pi^{\mu\nu}$



Conclusions

1. **Second-order fluid dynamics has been systematically derived as long-wavelength, small-frequency limit of kinetic theory**
2. **Transport coefficients agree with values from Chapman-Enskog expansion**
3. **Heat-flow problem can be solved by taking higher irreducible moments to be dynamical variables**
4. **Further systematic improvements are possible and should be explored**