Proposal for a running coupling JIMWLK equation

T. Lappi

University of Jyväskylä, Finland

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Outline

- CGC, Glasma, JIMWLK evolution
- JIMWLK equation in Langevin form
- Suggestion for a running coupling JIMWLK equaton
  T.L., H. Mäntysaari EPJC 2013

JIMWLK [“gym-walk”]:
Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

Gluon saturation, Glass and Glasma

Small x: the hadron/nucleus wavefunction is characterized by **saturation scale** $Q_s \gg \Lambda_{QCD}$.

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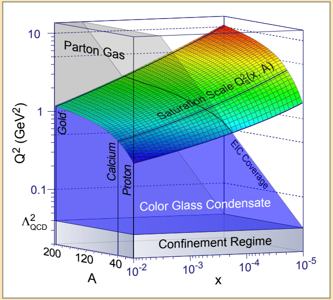
p $\sim Q_s$: strong fields $A_\mu \sim 1/g$

- occupation numbers $\sim 1/\alpha_s$
- classical field approximation.
- small α_s, but nonperturbative



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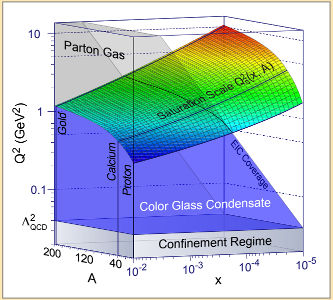

CGC: Effective theory for wavefunction of nucleus

- Large x = source ρ, **probability** distribution $W_y[\rho]$
- Small x = classical gluon field A_μ + quantum flucts.

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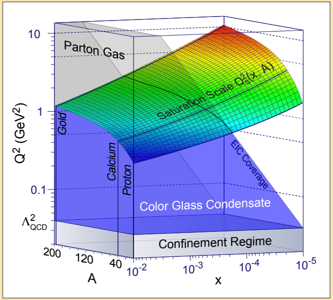

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Glasma field configuration of two colliding sheets of CGC.

Wilson line

Classical color field described as Wilson line

$$U(\mathbf{x}) = P \exp\left\{ ig \int \mathrm{d}x^- A^+_{\text{cov}}(\mathbf{x}, x^-) \right\} \quad \in \quad \text{SU}(3)$$

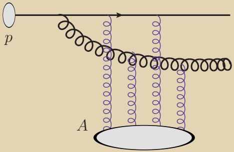
Relation to color charge

$$\nabla^2 A^+_{\text{cov}}(\mathbf{x}, x^-) = -g\rho(\mathbf{x}, x^-)$$

($x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$; $A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^z)$; $\mathbf{x}$ 2d transverse)

Example of usage: forward pA

- Quark from p (large x pdf) , radiate gluon
- Eikonal propagation $\Longrightarrow$ Wilson lines $U(\mathbf{x})$



Need target expectation values of operators:

$\text{Tr}\, U(\mathbf{x})U^\dagger(\mathbf{y}) \quad \text{Tr}\, U(\mathbf{x})U^\dagger(\mathbf{y})U(\mathbf{u})U^\dagger(\mathbf{v}) \quad \ldots$

JIMWLK evolution

Classical color field described as Wilson line

$$U(\mathbf{x}) = P\exp\left\{ig\int dx^- A^+(\mathbf{x}, x^-)\right\} \in SU(3)$$

- Energy dependent **probability** distribution $W_y[U]$ $(y \sim \ln\sqrt{s})$
- Energy/rapidity dependence of $W_y[U]$ from JIMWLK renormalization group equation

$$\partial_y W_y[U(\mathbf{x})] = \mathcal{H} W_y[U(\mathbf{x})]$$

JIMWLK Hamiltonian: (fixed coupling)

$$\mathcal{H} \equiv \frac{1}{2}\alpha_s \int_{\mathbf{xyz}} \frac{\delta}{\delta A_c^+(\mathbf{y})} \mathbf{e}^{ba}(\mathbf{x},\mathbf{z}) \cdot \mathbf{e}^{ca}(\mathbf{y},\mathbf{z}) \frac{\delta}{\delta A_b^+(\mathbf{x})},$$

$$\mathbf{e}^{ba}(\mathbf{x},\mathbf{z}) = \frac{1}{\sqrt{4\pi^3}} \frac{\mathbf{x}-\mathbf{z}}{(\mathbf{x}-\mathbf{z})^2} \left(1 - U^\dagger(\mathbf{x})U(\mathbf{z})\right)^{ba}$$

Fokker-Planck and Langevin

Textbook example: two descriptions of Brownian motion

- 1-d diffusion eq. ($\supset$ F.-P. eq.)

$$\partial_t P(x,t) = D\partial_x^2 P(x,t)$$

- $P(x,t)$=probability for particle to be at location x at time t.
- For particle starting at $x=0$ at $t=0$ solution is

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left\{-\frac{x^2}{4Dt}\right\}$$

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- Langevin equation:

$$\begin{aligned} x(t) &= \sqrt{2D}\eta(t) \\ \langle \eta(t)\eta(t')\rangle &= \delta(t-t') \end{aligned}$$

- $$\begin{aligned} \langle x(t)\rangle &= 0 \\ \langle x^2(t)\rangle &= 2Dt \end{aligned}$$

  $\Longrightarrow$ same as F.-P.

- Also easy to calculate $t \neq t'$

$$\langle x(t)x(t')\rangle = 2D\min(t,t')$$

- Now $x \Longrightarrow U(\mathbf{x})$ and $t \Longrightarrow y$.
- $({N_c}^2 - 1)N_\perp^2$-dimensional nonlinear diffusion equation. ($N_\perp^2$ = number of lattice points in transverse plane.)

Langevin formulation

Fokker-Planck $\Longrightarrow$ Langevin in JIMWLK Blaizot, Iancu, Weigert 2002

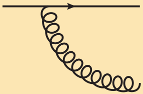
Original Langevin form: only right derivative ($\xi_{\mathbf{z}}^{b,i}$ is noise)

$$U_{\mathbf{x}}(y+\mathrm{d}y) = U_{\mathbf{x}}(y)\exp\left\{ it^a \int_{\mathbf{z}} \varepsilon_{\mathbf{x},\mathbf{z}}^{ab,i}\xi_{\mathbf{z}}^{b,i}\sqrt{\mathrm{d}y} + \sigma_{\mathbf{x}}^a \mathrm{d}y \right\}.$$

New simpler, equivalent (for $\mathrm{d}y \to 0$) form T.L., H.M.

$$U_{\mathbf{x}}(y+\mathrm{d}y) = \exp\left\{ -i\frac{\sqrt{\alpha_s \,\mathrm{d}y}}{\pi}\int_{\mathbf{z}} \mathbf{K}_{\mathbf{x}-\mathbf{z}} \cdot (U_{\mathbf{z}}\xi_{\mathbf{z}}U_{\mathbf{z}}^\dagger) \right\}$$
$$\times\, U_{\mathbf{x}}(y) \exp\left\{ i\frac{\sqrt{\alpha_s\,\mathrm{d}y}}{\pi}\int_{\mathbf{z}} \mathbf{K}_{\mathbf{x}-\mathbf{z}}\cdot\xi_{\mathbf{z}} \right\},$$

$$K_{\mathbf{x}-\mathbf{z}}^i = \frac{(\mathbf{x}-\mathbf{z})^i}{(\mathbf{x}-\mathbf{z})^2}$$



$i = x, y$

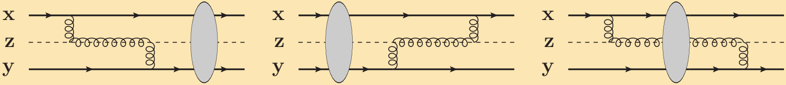
Fixed α_s noise: $\langle \xi_{\mathbf{x}}(y_m)_i^a \xi_{\mathbf{y}}(y_n)_j^b \rangle = \alpha_s \delta^{ab}\delta^{ij}\delta_{\mathbf{xy}}^{(2)}\delta_{mn}, \quad \xi = \xi^a t^a$

Multiply from left **and** right $\Longrightarrow$ no deterministic term

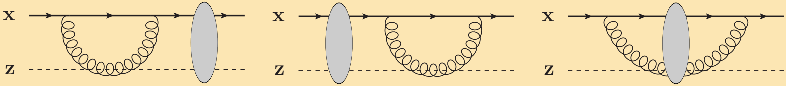
Interpreting JIMWLK: derive BK

$$U_{\mathbf{x}}(y+\mathrm{d}y) = e^{-i\frac{\sqrt{\alpha_s\,\mathrm{d}y}}{\pi}\int_{\mathbf{z}} \mathbf{K}_{\mathbf{x}-\mathbf{z}}\cdot(U_{\mathbf{z}}\xi_{\mathbf{z}}U^\dagger_{\mathbf{z}})}\, U_{\mathbf{x}}\, e^{i\frac{\sqrt{\alpha_s\,\mathrm{d}y}}{\pi}\int_{\mathbf{z}} \mathbf{K}_{\mathbf{x}-\mathbf{z}}\cdot\xi_{\mathbf{z}}},$$

- At $\mathrm{d}y \to 0$ develop to $\mathcal{O}(\xi^2)$ and take expectation values.
- BK Balitsky-Kovchegov is equation for **dipole** $\hat{D}_{\mathbf{x},\mathbf{y}} = \mathrm{Tr}\, U^\dagger(\mathbf{x})U(\mathbf{y})/N_c$
- Contract ξ's from timestep of $U^\dagger(\mathbf{x})$ with one from $U(\mathbf{y})$: real terms



- Contract two ξ's from timestep of $U^\dagger(\mathbf{x})$ or $U(\mathbf{y})$: virtual terms



- Result

$$\partial_y \hat{D}_{\mathbf{x},\mathbf{y}}(y) = \frac{\alpha_s N_c}{2\pi^2}\int_{\mathbf{z}} \left(\mathbf{K}^2_{\mathbf{x}-\mathbf{z}} + \mathbf{K}^2_{\mathbf{y}-\mathbf{z}} - 2\mathbf{K}_{\mathbf{x}-\mathbf{z}}\cdot\mathbf{K}_{\mathbf{y}-\mathbf{z}}\right)\left[\hat{D}_{\mathbf{x},\mathbf{z}}\hat{D}_{\mathbf{z},\mathbf{y}} - \hat{D}_{\mathbf{x},\mathbf{y}}\right].$$

Scale of running α_s in JIMWLK

BK for $\hat{D}_{\mathbf{x},\mathbf{y}}(y)$ describes dipole splitting $\mathbf{x}-\mathbf{y} \quad \longrightarrow \quad \mathbf{x}-\mathbf{z}\,;\ \mathbf{z}-\mathbf{y}$

- α_s given by parent $\mathbf{x}-\mathbf{y}$: easy in BK, but funny in JIMWLK: Langevin is only for one Wilson line
- Daughter (scale in $\mathbf{K}$): easy to implement as $\sqrt{\alpha_s}$, but why?

$$\sqrt{\alpha_s}\mathbf{K}_{\mathbf{x}-\mathbf{z}} \to \sqrt{\alpha_s(\mathbf{x}-\mathbf{z})}\mathbf{K}_{\mathbf{x}-\mathbf{z}}$$

- Used in BK: combinations of these two.
- Suggestion T.L., H.Mäntysaari 2012 : natural scale is momentum of radiated gluon.
- Implemented by modifying momentum space noise correlator

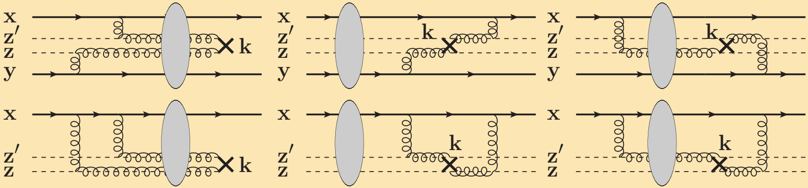
$$\langle \xi_{\mathbf{x}}(m)_i^a \xi_{\mathbf{y}}(n)_j^b \rangle \sim \alpha_s \delta^{(2)}_{\mathbf{x}\mathbf{y}} = \alpha_s \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$\Longrightarrow \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \alpha_s(\mathbf{k})$$

Reinterpreting JIMWLK

$$U_{\mathbf{x}}(y+\mathrm{d}y) = \exp\left\{-i\frac{\sqrt{\mathrm{d}y}}{\pi}\int_{\mathbf{z}} \mathbf{K}_{\mathbf{x}-\mathbf{z}}\cdot\left(U_{\mathbf{z}}\xi_{\mathbf{z}}U^{\dagger}_{\mathbf{z}}\right)\right\} \times U_{\mathbf{x}}(y)\ \exp\left\{i\frac{\sqrt{\mathrm{d}y}}{\pi}\int_{\mathbf{z}'} \mathbf{K}_{\mathbf{x}-\mathbf{z}'}\cdot\xi_{\mathbf{z}'}\right\},$$

$$\langle \xi_{\mathbf{x}}(m)^a_i \xi_{\mathbf{y}}(n)^b_j\rangle \sim \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\alpha_{\mathrm{s}}(\mathbf{k}) \equiv \widetilde{\alpha}_{\mathbf{x}-\mathbf{y}}$$



- Breaks time-reversal-symmetry: choose scale as momentum of gluon either before or after the target
- Two gluon coordinates instead of one

Recovering BK

- Equation for dipole now involves higher point functions:

$$\partial_y \hat{D} = \frac{N_c}{2\pi^2} \int_{\mathbf{u},\mathbf{v}} \widetilde{\alpha}_{\mathbf{u}-\mathbf{v}} \Big(\mathbf{K}_{\mathbf{x}-\mathbf{u}} \cdot \mathbf{K}_{\mathbf{x}-\mathbf{v}} + \mathbf{K}_{\mathbf{y}-\mathbf{u}} \cdot \mathbf{K}_{\mathbf{y}-\mathbf{v}} - 2\mathbf{K}_{\mathbf{x}-\mathbf{u}} \cdot \mathbf{K}_{\mathbf{y}-\mathbf{v}} \Big) \times \frac{1}{2} \left[\hat{D}_{\mathbf{x},\mathbf{u}} \hat{D}_{\mathbf{u},\mathbf{y}} + \hat{D}_{\mathbf{x},\mathbf{v}} \hat{D}_{\mathbf{v},\mathbf{y}} - \hat{D}_{\mathbf{x},\mathbf{y}} - \hat{D}_{\mathbf{v},\mathbf{u}} \hat{Q}_{\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{y}} \right],$$

- But recall that α_s is a slowly varying function of the scale:

$$\widetilde{\alpha}_{\mathbf{x}-\mathbf{y}} \equiv \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \alpha_s(\mathbf{k}) \sim \alpha_s \delta^2(\mathbf{x}-\mathbf{y})$$

$\Longrightarrow$ $\mathbf{u} \approx \mathbf{v}$ and structure simplifies to BK:

$$\frac{1}{2} \left[\hat{D}_{\mathbf{x},\mathbf{u}} \hat{D}_{\mathbf{u},\mathbf{y}} + \hat{D}_{\mathbf{x},\mathbf{v}} \hat{D}_{\mathbf{v},\mathbf{y}} - \hat{D}_{\mathbf{x},\mathbf{y}} - \hat{D}_{\mathbf{v},\mathbf{u}} \hat{Q}_{\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{y}} \right] \approx \hat{D}_{\mathbf{x},\mathbf{u}} \hat{D}_{\mathbf{u},\mathbf{y}} - \hat{D}_{\mathbf{x},\mathbf{y}}$$

- Parametrically dominant length scale in coupling is “smallest dipole”, just like in Balitsky prescription for BK.

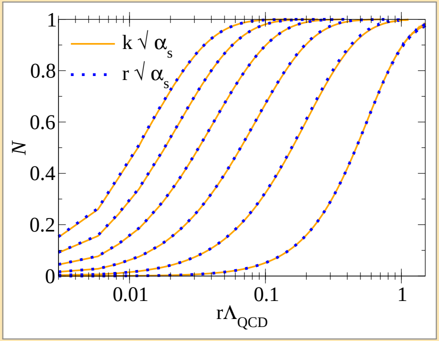
Side note: scale in coordinate vs momentum space

If running coupling depends only on scale in **K** ($\sqrt{\alpha_s}$-prescription), can use either coordinate or momentum space:

$$\sqrt{\alpha_s(\mathbf{r})}\frac{\mathbf{r}}{\mathbf{r}^2}$$

vs.

$$\sqrt{\alpha_s(\mathbf{k})}\frac{\mathbf{k}}{\mathbf{k}^2}$$

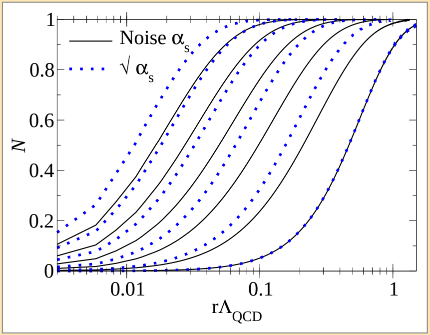


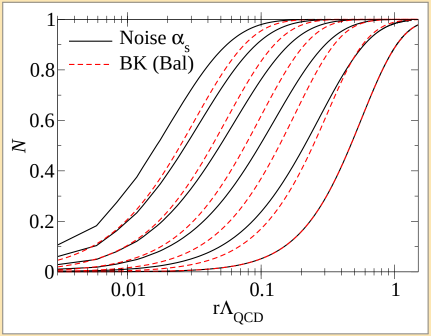
Numerically verified identification Kovchegov, Weigert **for this kernel**

$$\ln\frac{\mathbf{k}^2}{\Lambda^2_{\mathrm{QCD}}} \sim \ln\frac{4e^{-2\gamma_E}}{\mathbf{r}^2\Lambda^2_{\mathrm{QCD}}}$$

In numerical comparisons we assume this identification generally.

Comparison BK/JIMWLK





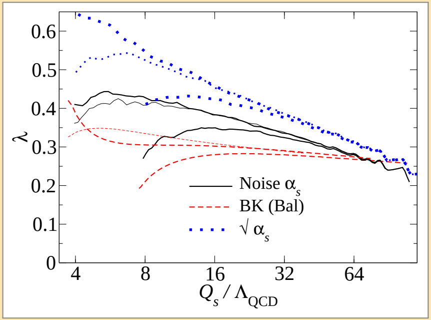
Evolution with our prescription is slower than with $\sqrt{\alpha_s}$.
This is good, data favors slower evolution

But this is still faster than with Balitsky prescription in BK
(Although parametricallly dominant scales are the same.)

Note: rcBK fits to HERA data need to take $\Lambda_{QCD} \approx 50$MeV to make evolution slow enough.

Evolution speed

$$\lambda \equiv \frac{d \ln Q_s^2}{dy}$$



- At very small Q_s also dependence on how the Landau pole is regulated (different line shapes)
- At very large Q_s lattice UV cutoff slows down JIMWLK simulations

Conclusions

- JIMWLK equation is beginning to be actually applied
- Running coupling, going towards NLO . . . necessary for phenomenology
- Provides initial state for AA collision in CYM