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# Perturbative QCD

## from basic principles to current applications

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disclaimer:

pQCD is about 40 years old – impossible to review in 3 hrs

# topics & questions to be addressed

**we will mainly concentrate on a few basics**
**and their consequences for phenomenology**

- What are the foundations of QCD?
  keywords: color; SU(3) gauge group; local gauge invariance; Feynman rules
- What are the general features of QCD?
  keywords: asymptotic freedom; infrared safety; origin of "singularities"
- How to relate QCD to experiment?
  keywords: partons; factorization; renormalization group eqs. / evolution
- How reliable is a theoretical QCD calculation?
  keywords: scale dependence; NLO; small-x; all-order resummations
- What is the status of some non-perturbative inputs
  keywords: global QCD analysis

# bibliography – a personal selection

Photo by Matt Heyssler

## textbooks:

- the "pink book" on QCD and Collider Physics by R.K. Ellis, W.J. Stirling, and B.R. Webber always a good reference
- R.D. Field, Applications of pQCD detailed examples
- Y.V. Kovchegov, E. Levin, QCD at High Energy focus on small x physics
- J. Collins, Foundations of pQCD focus on formal aspects of evolution

## lecture notes & write-ups:

- D. Soper, Basics of QCD Perturbation Theory, hep-ph/9702203
- Collins, Soper, Sterman, Factorization of Hard Processes in QCD, hep-ph/0409313
- G. Salam, Elements of QCD for Hadron Colliders, arXiv:1011.5131
- Particle Data Group, Review of Particle Physics, pdg.lbl.gov

## talks & lectures on the web:

- annual CTEQ summer school, tons of material on www.cteq.org
- annual CERN/FNAL Hadron Collider Physics School hcpss.web.cern.ch/hcpss

# tentative outline of the lectures

**Part 1:** **the foundations**

SU(3); color algebra; gauge invariance;
QCD Lagrangian; Feynman rules

**Part 2:** **the QCD toolbox**

asymptotic freedom; infrared safety;
the QCD final-state; jets; factorization

**Part 3:** **inward bound: "femto spectroscopy"**

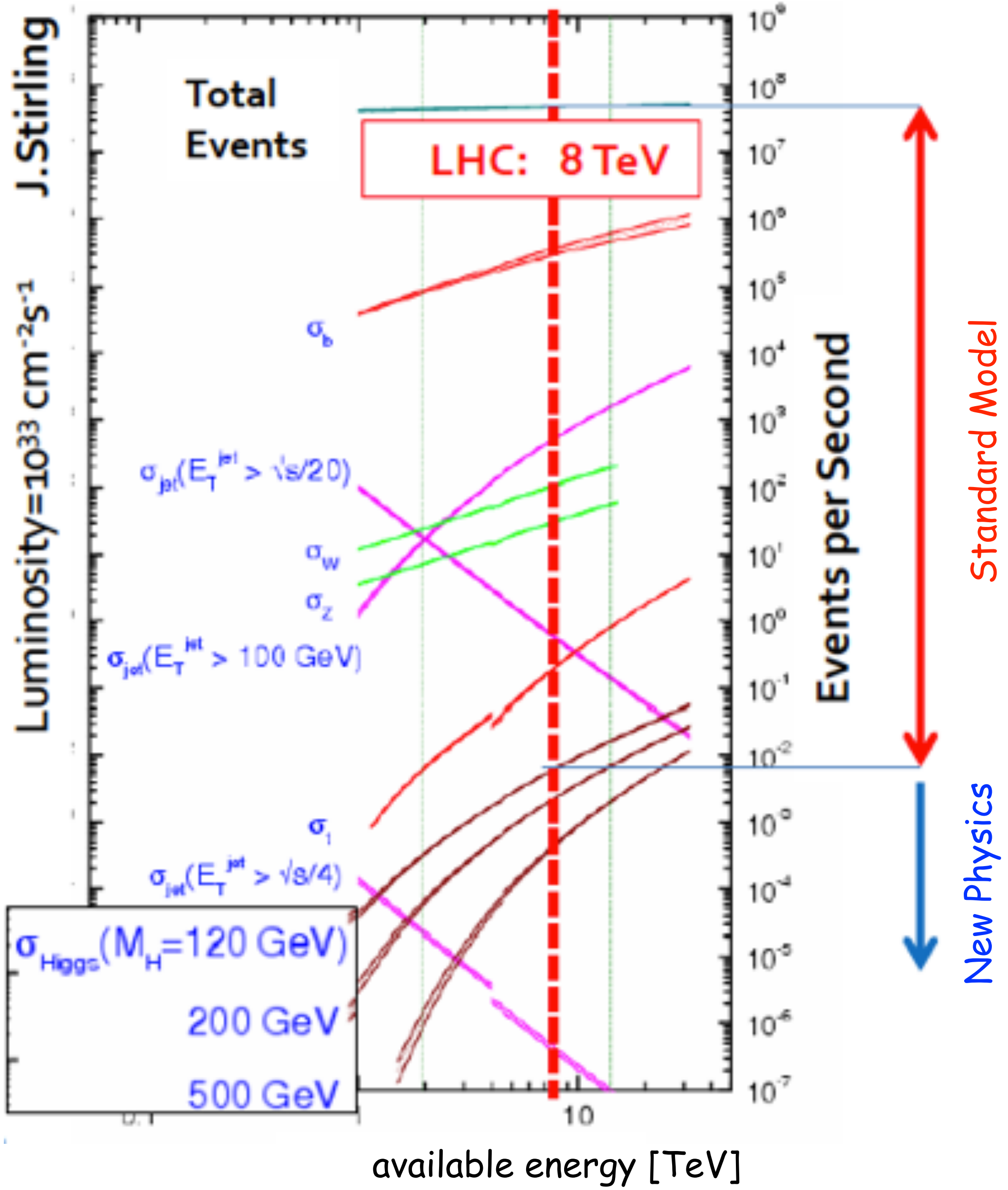
QCD initial-state; DIS process; partons;
factorization; renormalization group; scales;
hadron-hadron collisions

# Part I

## the QCD fundamentals

all about color
the concept of gauge invariance

# QCD – why do we still care (or perhaps more than ever)



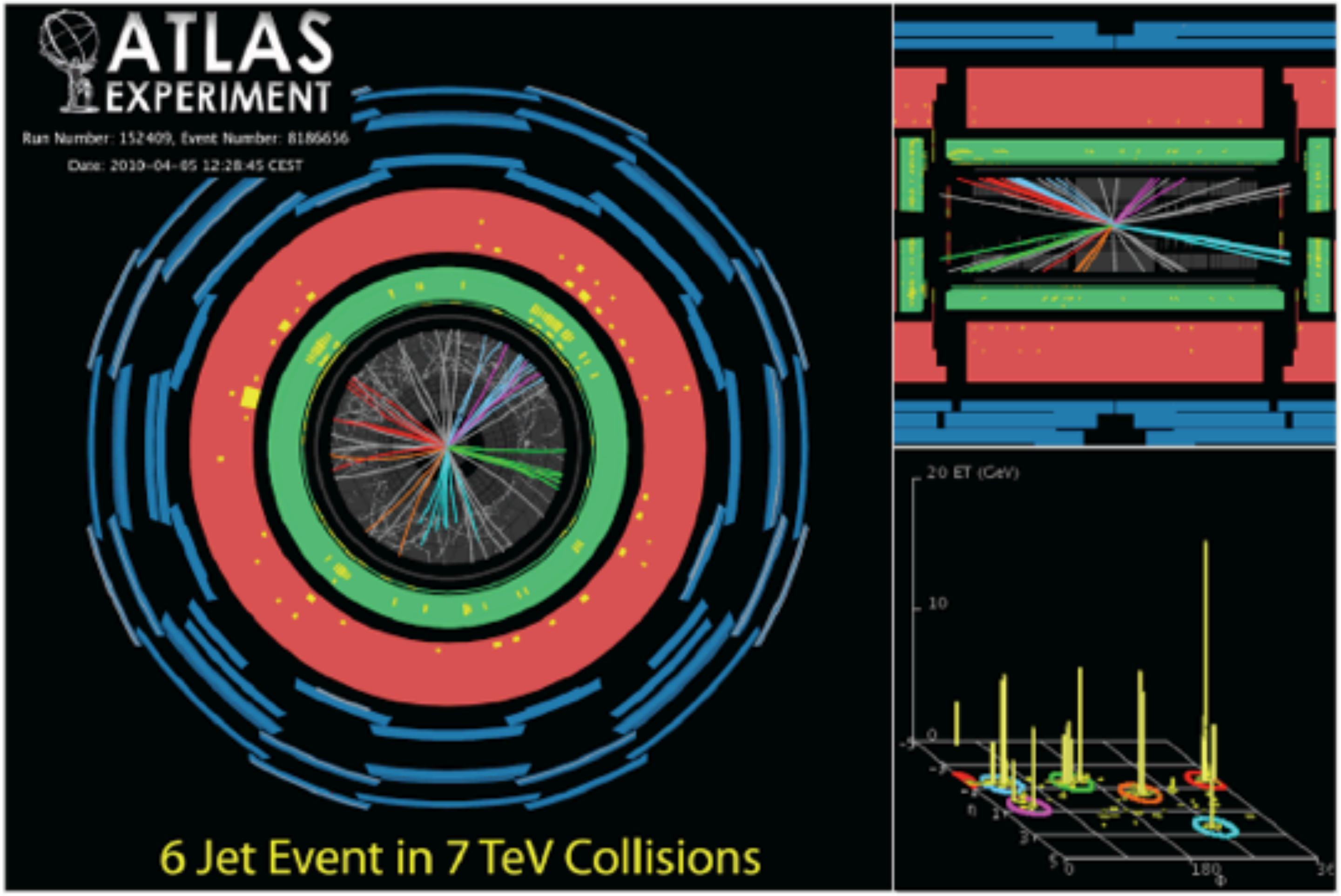
hadron colliders inevitably have to deal with QCD

discovering the Higgs or some New Physics requires a sophisticated quantitative understanding of QCD

P.W. Higgs, F. Englert (2013)

achieving that can be quite a challenge ...

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^{A}_{\mu\nu}F^{\mu\nu}_{A} + \sum_{\mathrm{flavors}} \bar{q}_i \left(i\not{D} - m\right)_{ij} q_j$$



# QCD – the theory of strong interactions

a simple QED-like theory, leading to extremely rich & complex phenomena

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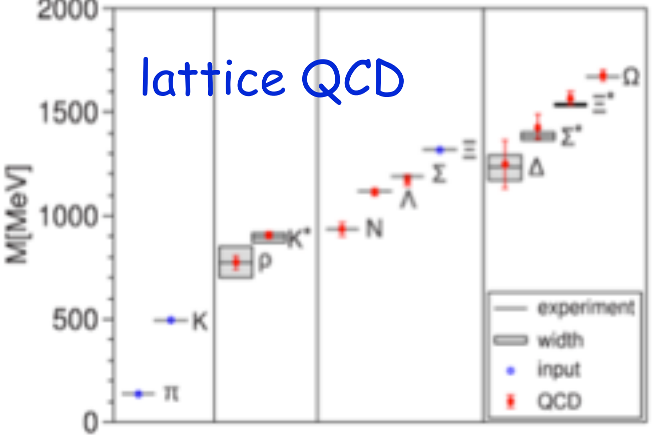
a simple QED-like theory, leading to extremely rich & complex phenomena



Durr et al., Science 322

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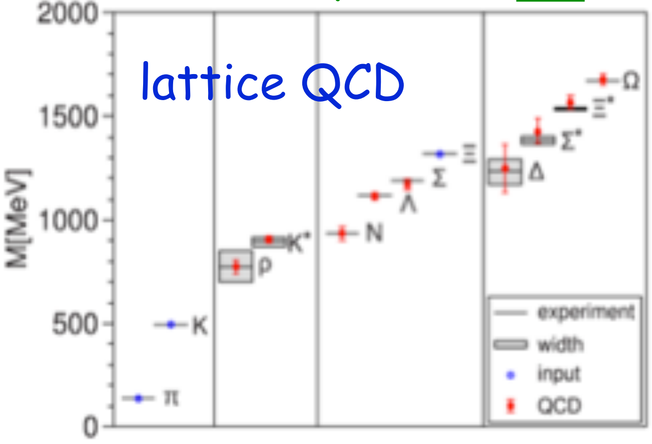
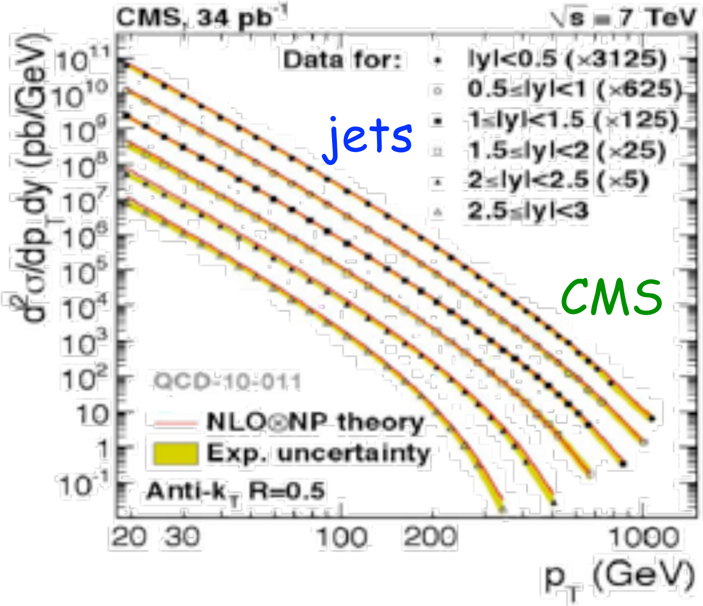

lattice QCD


jets

CMS

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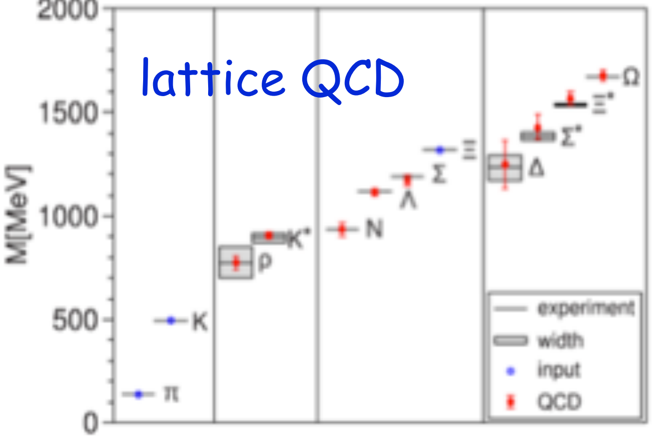
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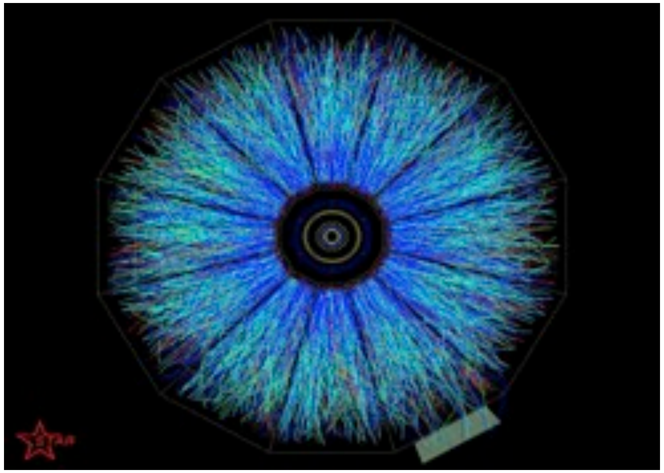
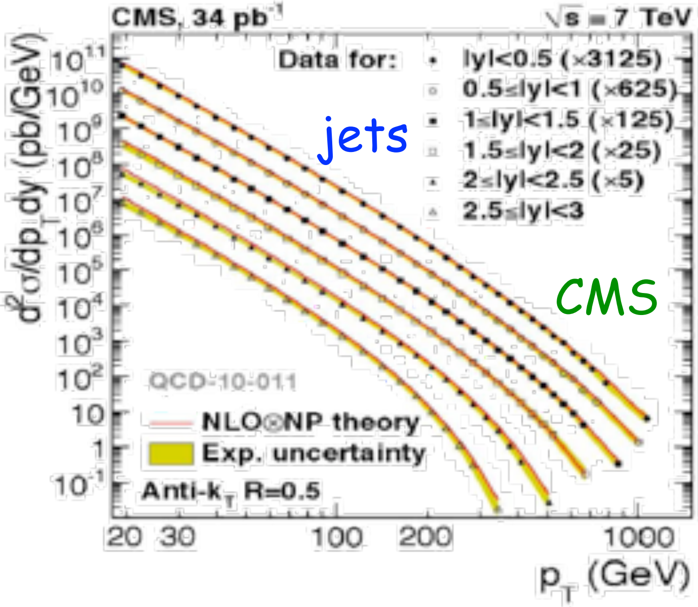


AuAu collision at STAR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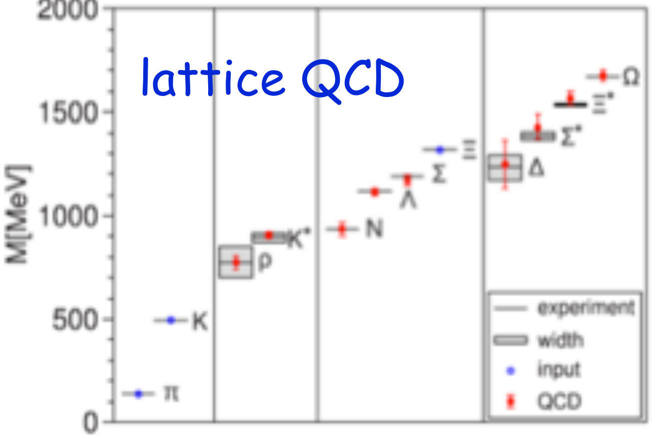

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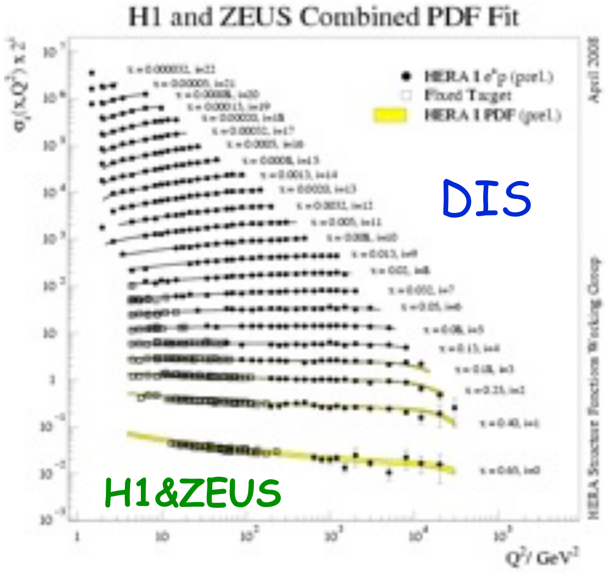
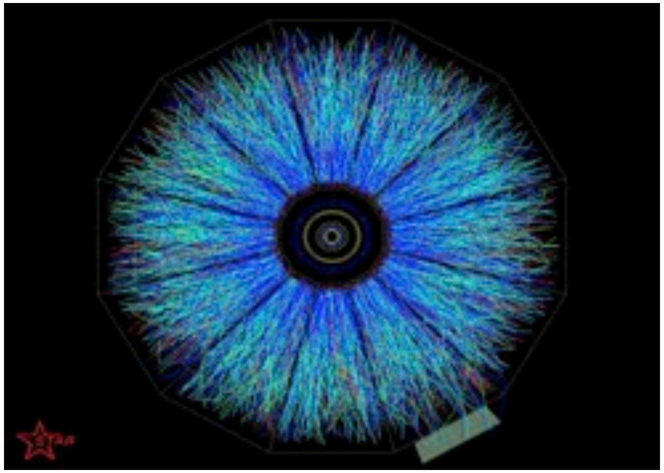


DIS

H1&ZEUS



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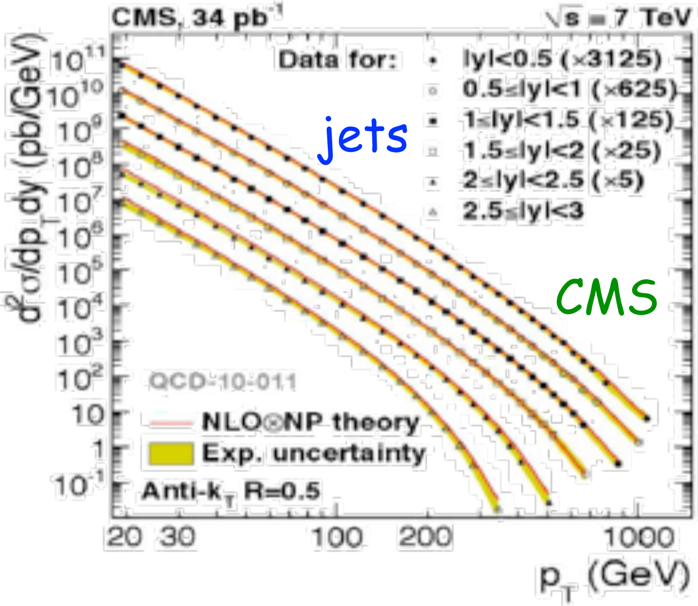

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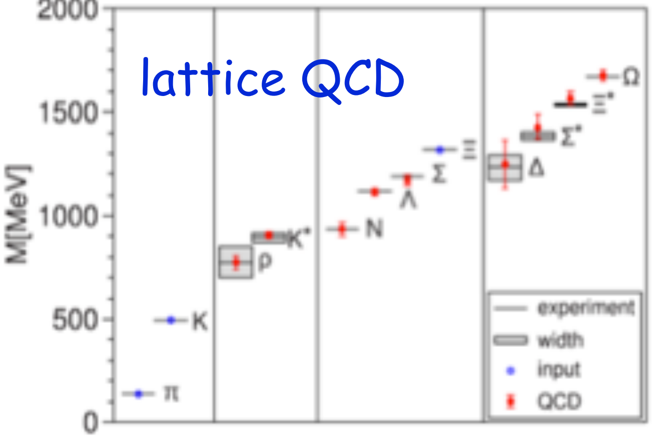

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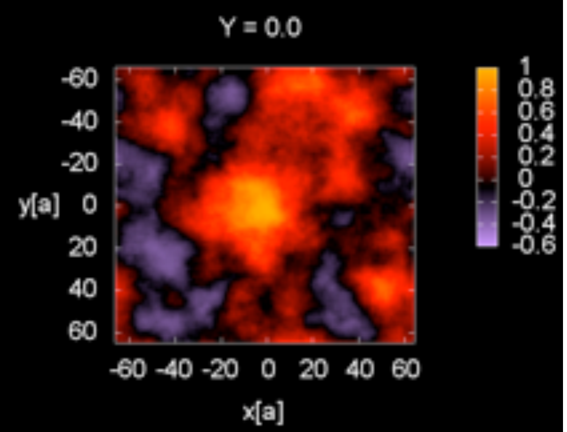
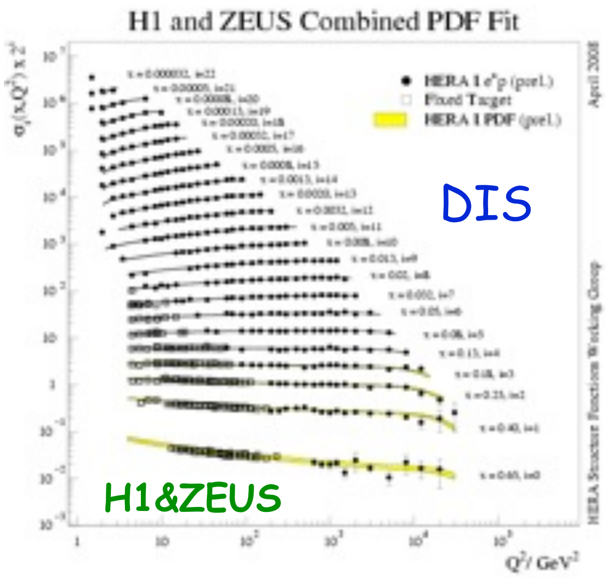


RGE evolution of gluon correlators
arXiv:1108.4764



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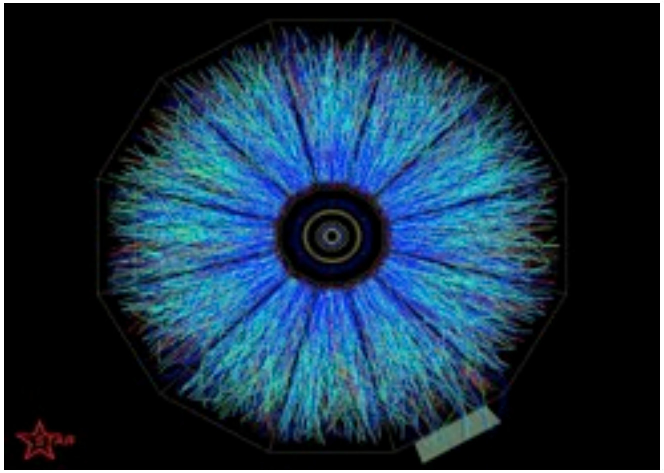

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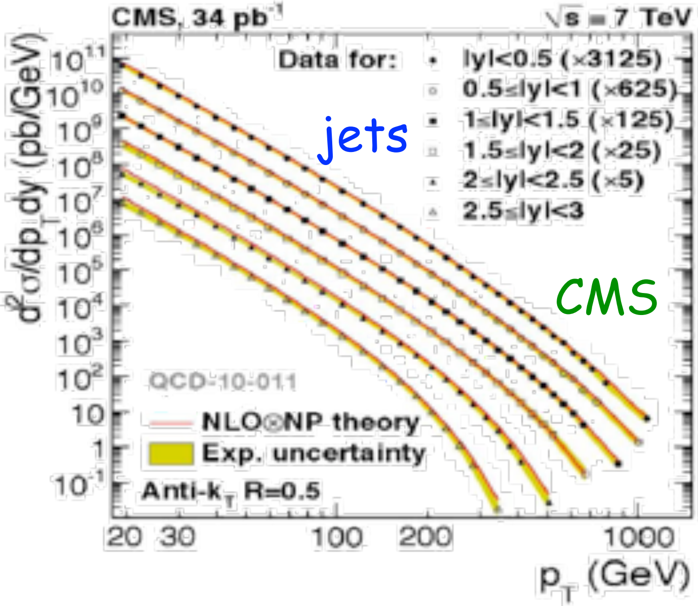

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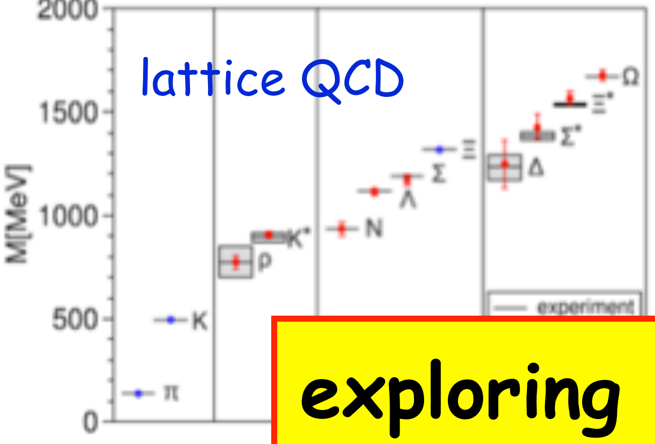

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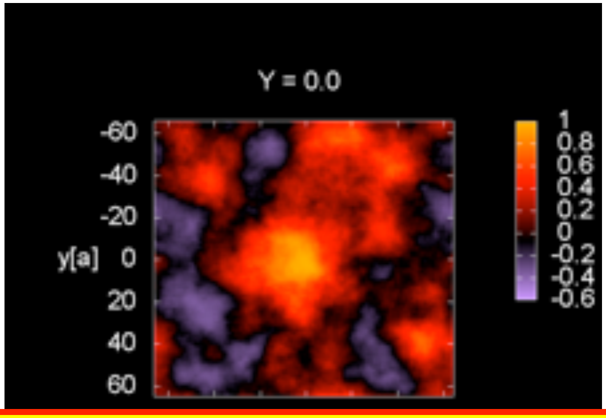
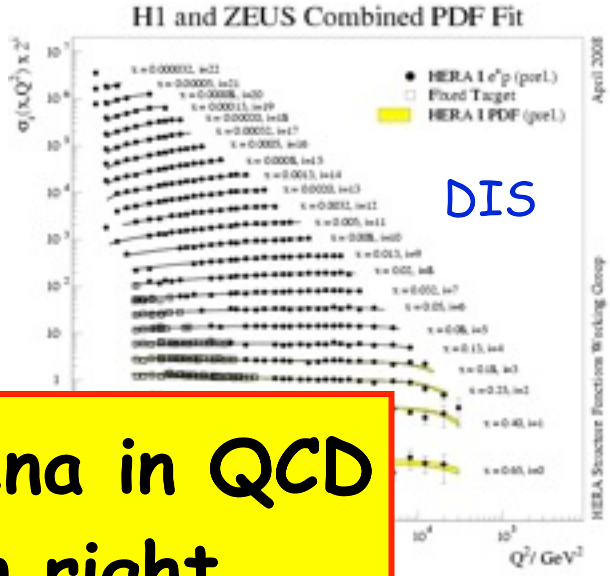
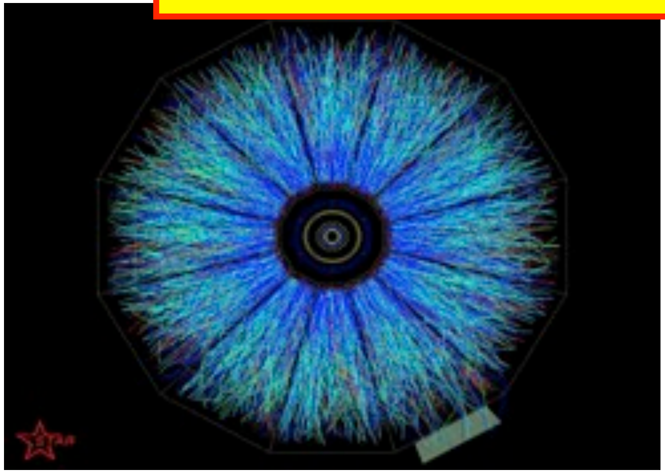




DIS

**exploring all these phenomena in QCD is interesting in its own right**



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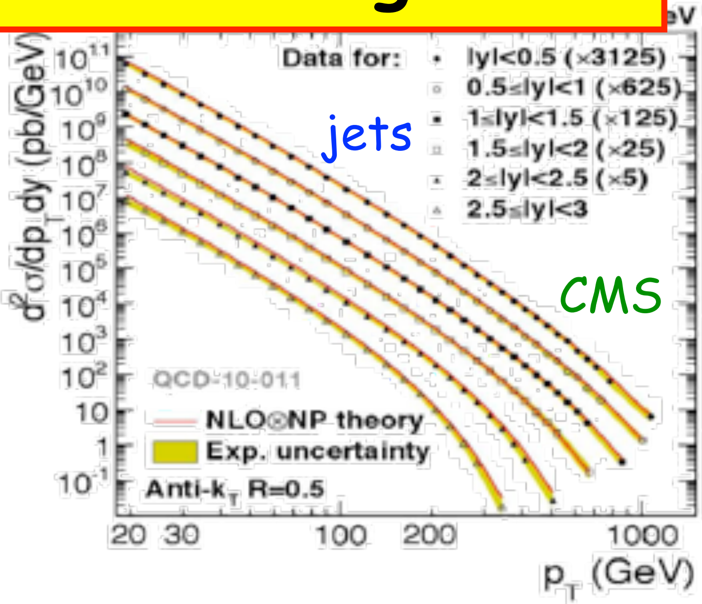

jets

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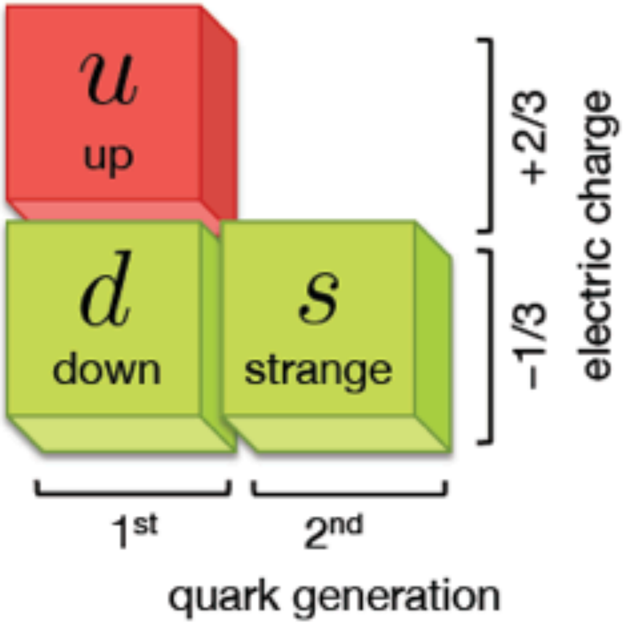
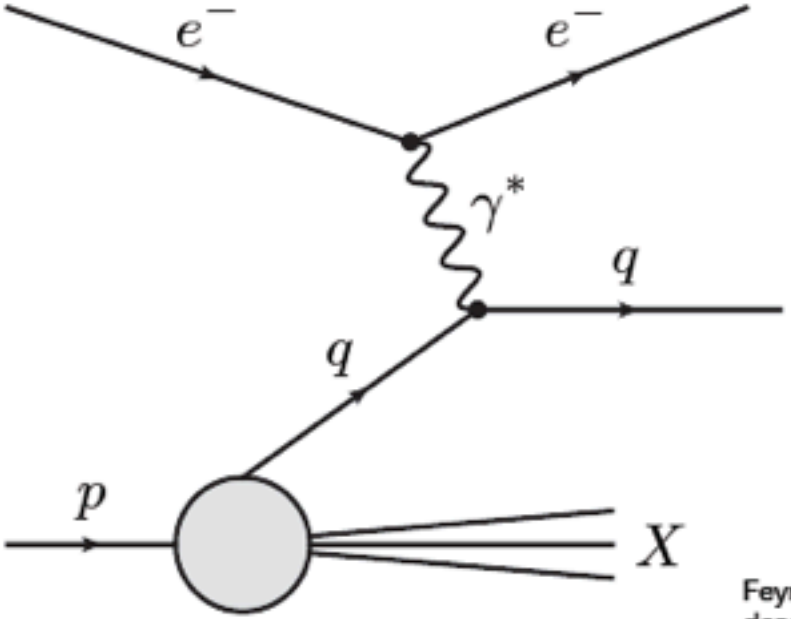
# QCD matter sector: Three Quarks for Muster Mark





Feynman diagram describing DIS of an electron on a proton

existence of light quarks validated in deep-inelastic scattering (DIS) experiments carried out at SLAC in 1968

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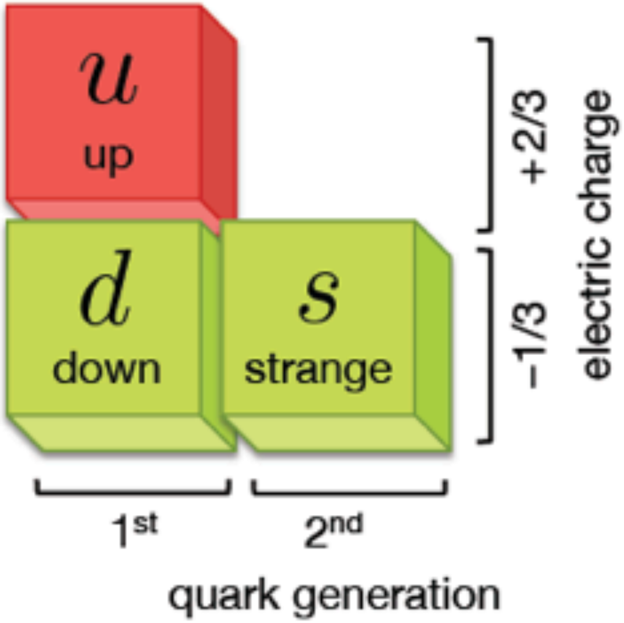
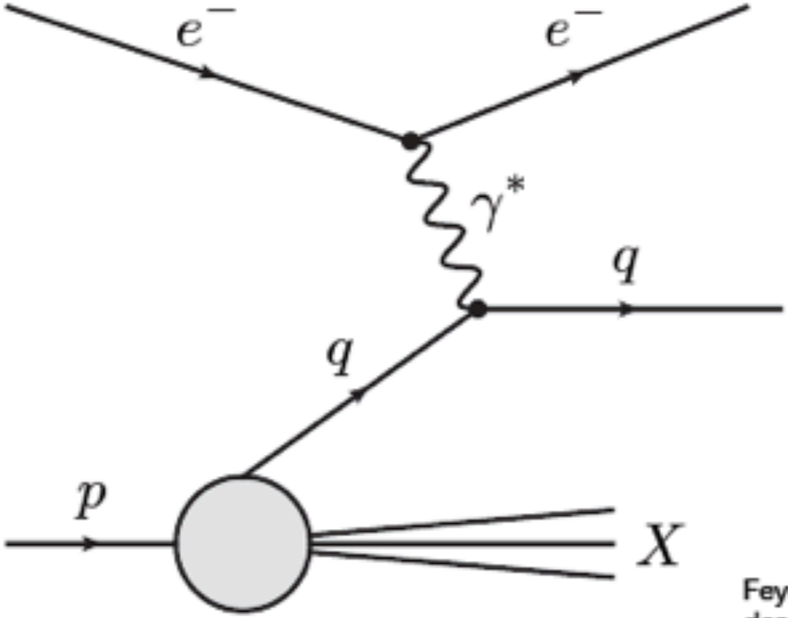




Feynman diagram describing DIS of an electron on a proton

existence of light quarks validated in deep-inelastic scattering (DIS) experiments carried out at SLAC in 1968

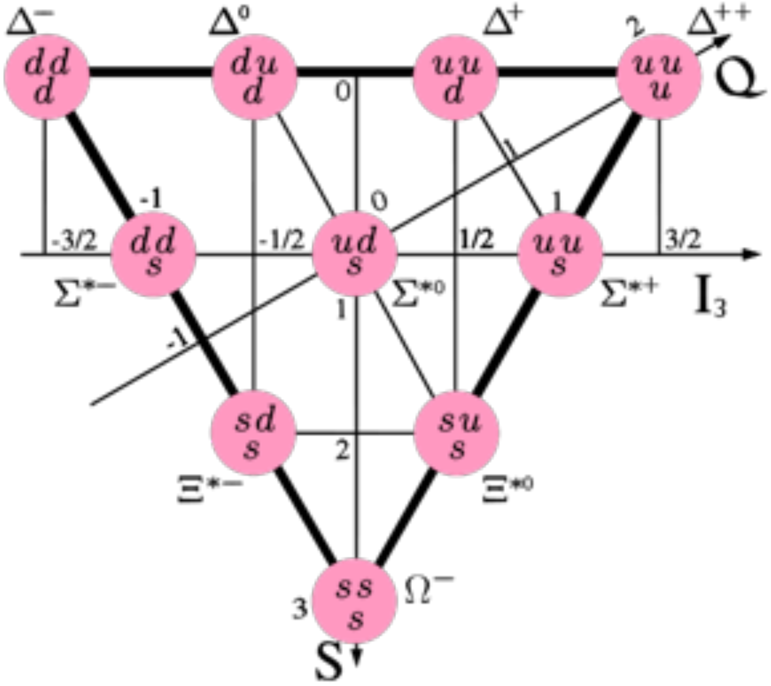
strange quarks necessary component in **quark model** to classify the observed slew of mesons/baryons Gell-Mann, Zweig (1964)

based on "**Eightfold Way**" (= $SU(3)_{flavor}$) Gell-Mann; Ne'eman (1961)

# quark model: mesons and baryons

categorizes mesons (baryons) in terms of two (three) constituent quarks in **SU(3)$_{\text{flavor}}$ multiplets** = octets and decuplets

baryon decuplet



spectrum fully classified by assuming:

- quarks have spin $\frac{1}{2}$
- quarks have fractional charges

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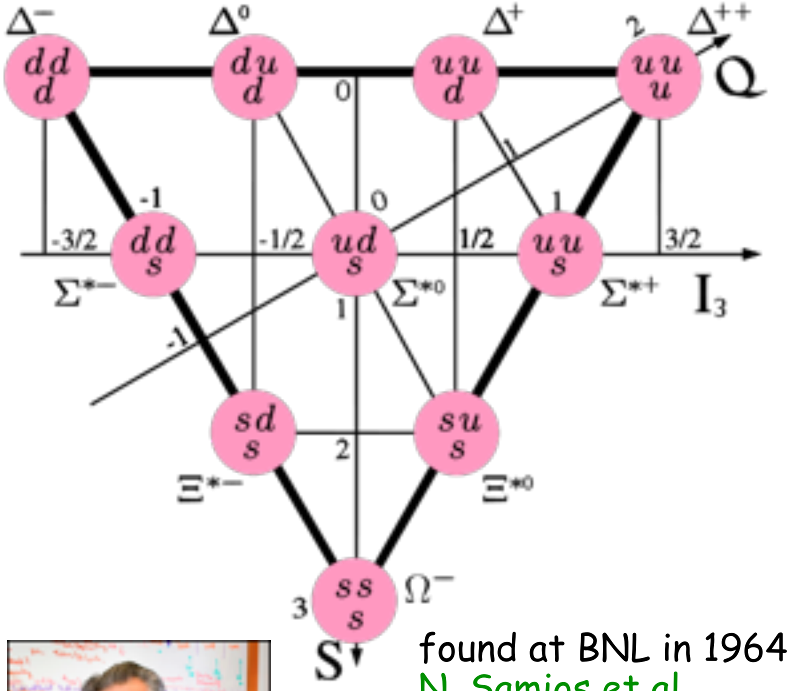

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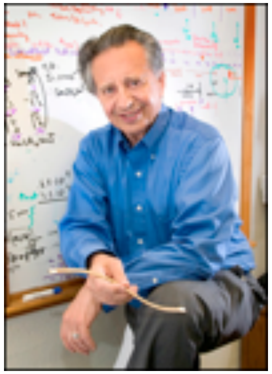
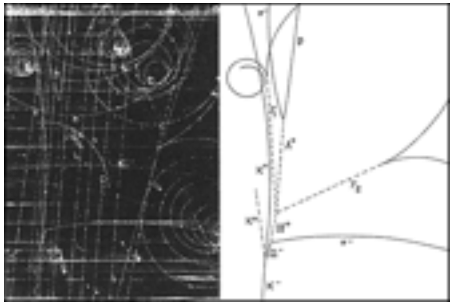
**big success:** prediction of $\Omega^-$ (sss)



found at BNL in 1964
N. Samios et al.



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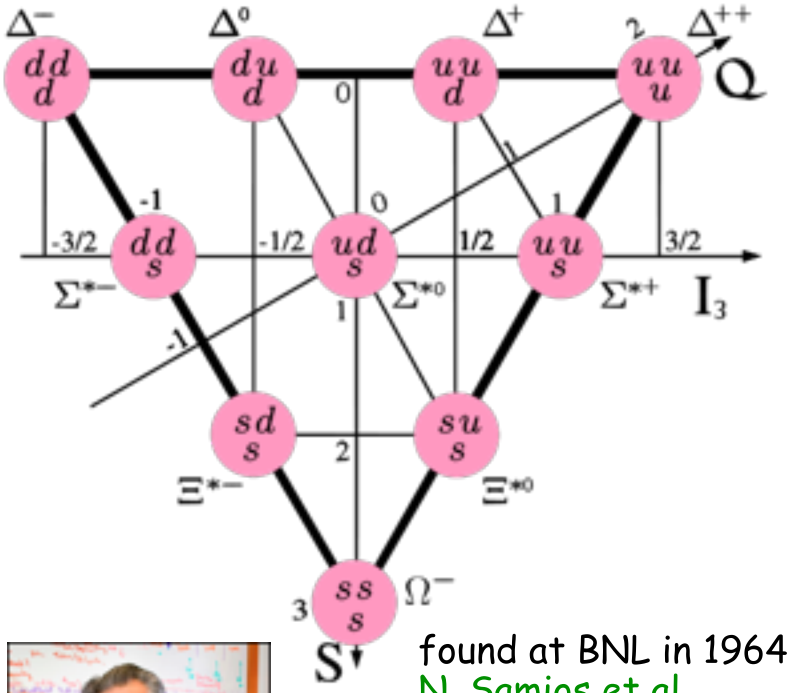

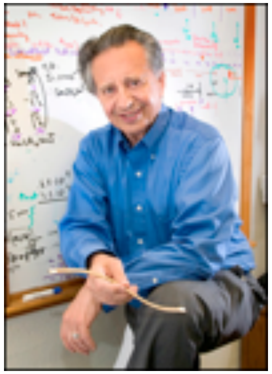

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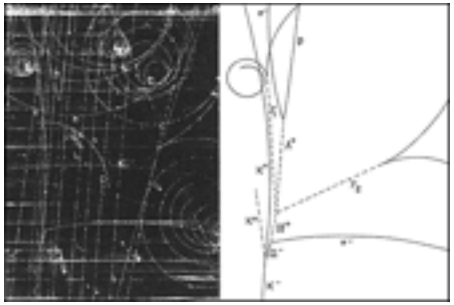

spectrum fully classified by assuming:

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  (but combine into hadrons with integer charges)

big success: prediction of $\Omega^-$ (sss)

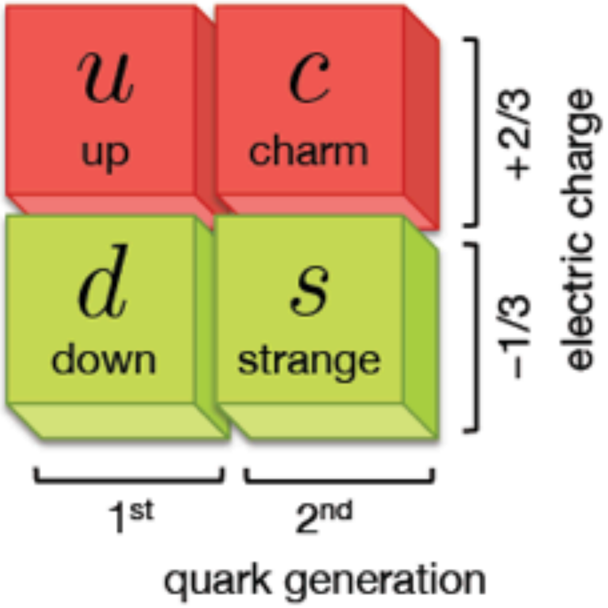
also, first evidence of color

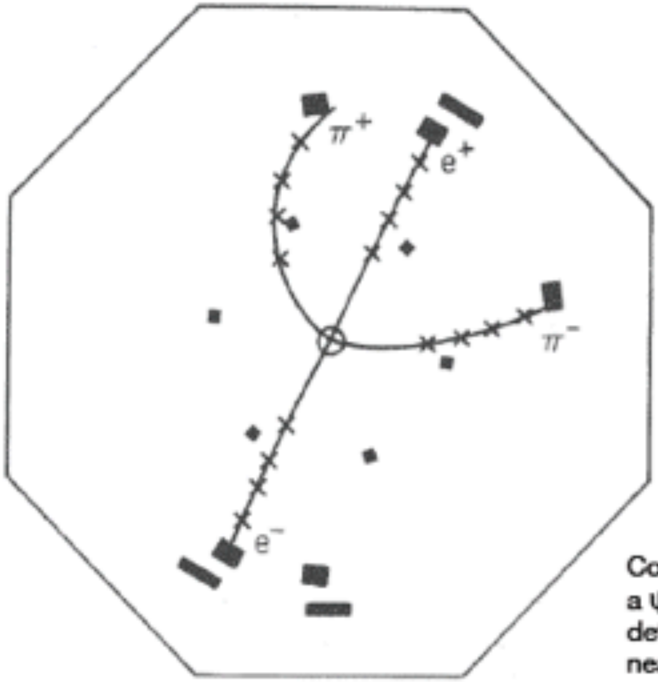
- $\Delta^{++}$ wave function |uuu> not anti-sym
  (violates Pauli principle)
- remedy: color quantum number
  but hadrons remain colorless/color singlets

$$\sim \sum_{ijk} \epsilon_{ijk} |q_i q_j q_k\rangle$$

$$\sim \sum_{i} |\bar{q}_i q_i\rangle$$

# QCD matter sector: charm





Computer reconstruction of a ψ′ decay in the Mark I detector at SLAC, making a near-perfect image of the Greek letter ψ

predicted on strong theoretical grounds (suppression of FCNC)

“GIM mechanism” in 1970 Glashow, Iliopolus, Maiani

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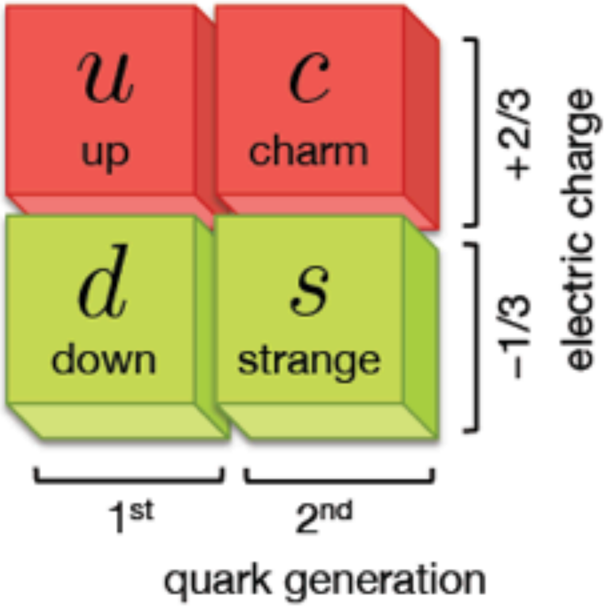

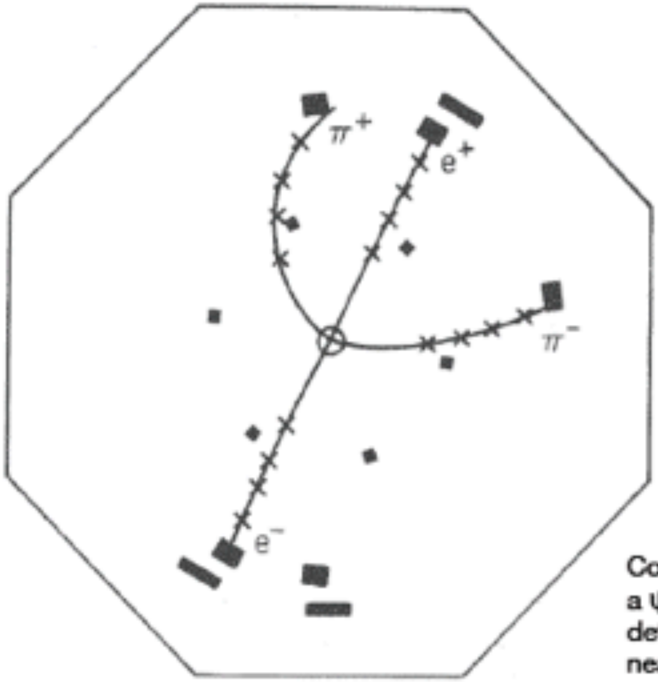

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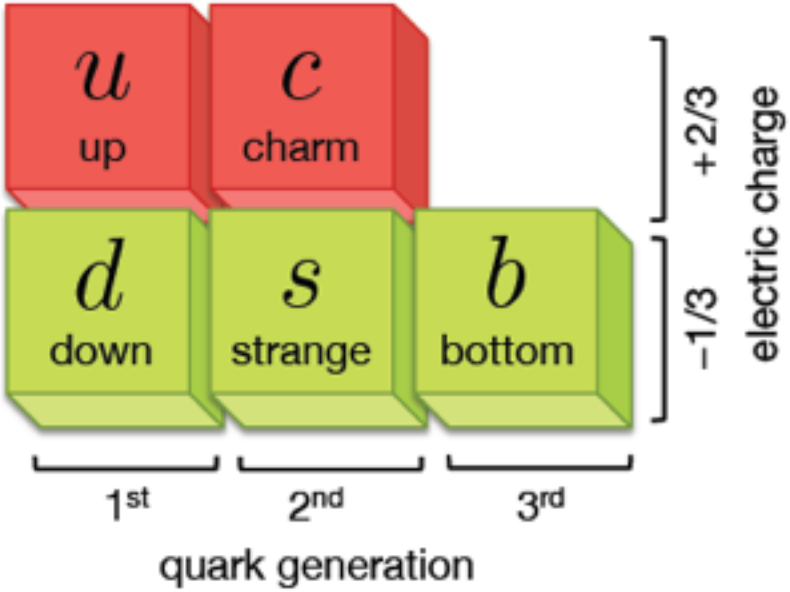
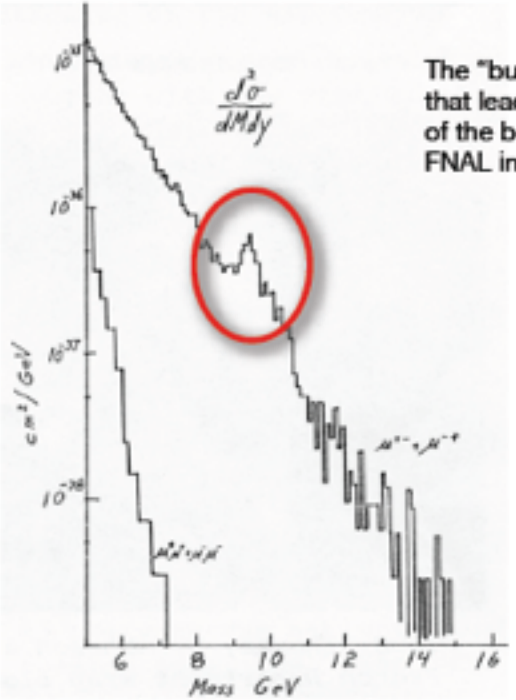
predicted on strong theoretical grounds (suppression of FCNC)
"GIM mechanism" in 1970 Glashow, Iliopolus, Maiani

observed during "November revolution" in 1974 both at SLAC (Richter et al.) and BNL (Ting et al.)
discovered meson became known as J/Ψ; Nobel Prize in 1976

# QCD matter sector: bottom





The "bump" at 9.5 GeV that lead to the discovery of the bottom quark at FNAL in 1977

theorized in 1973 in order to accommodate CP violation

(requires third generation)

Kobayashi, Maskawa Nobel Prize 2008

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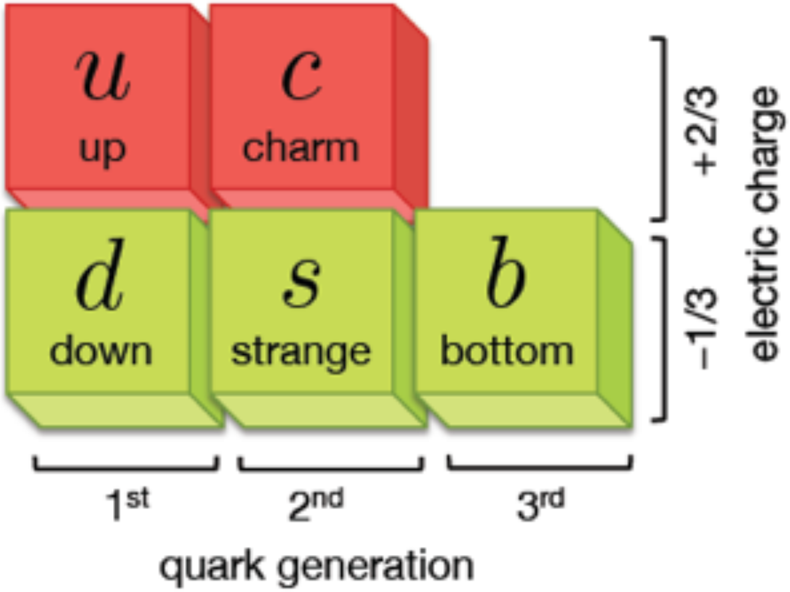

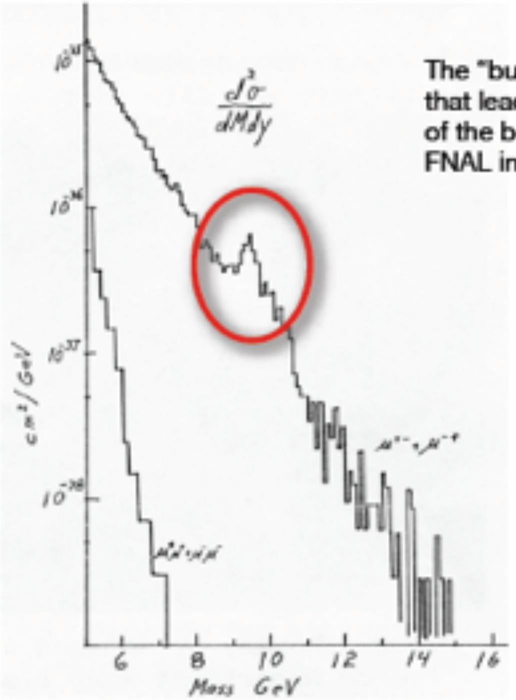

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Kobayashi, Maskawa Nobel Prize 2008

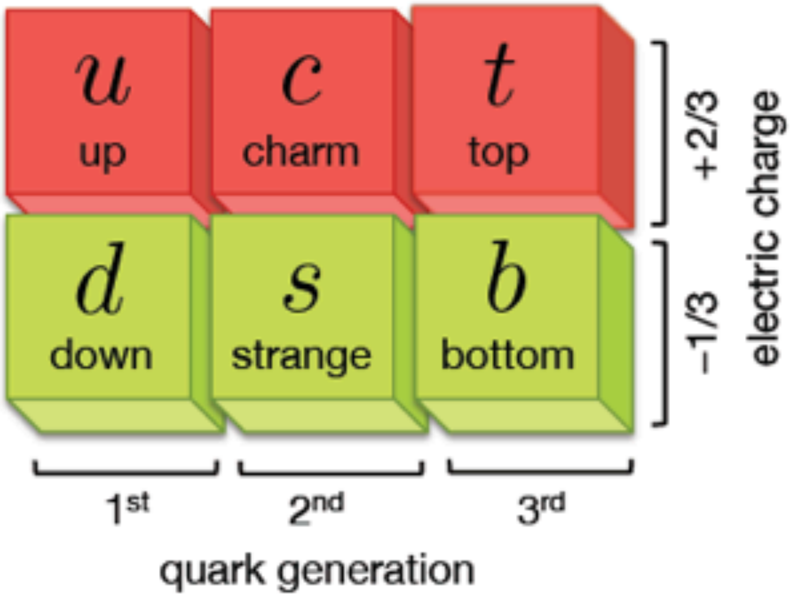
discovered in 1977 at FNAL (ϒ meson or "bottomium")

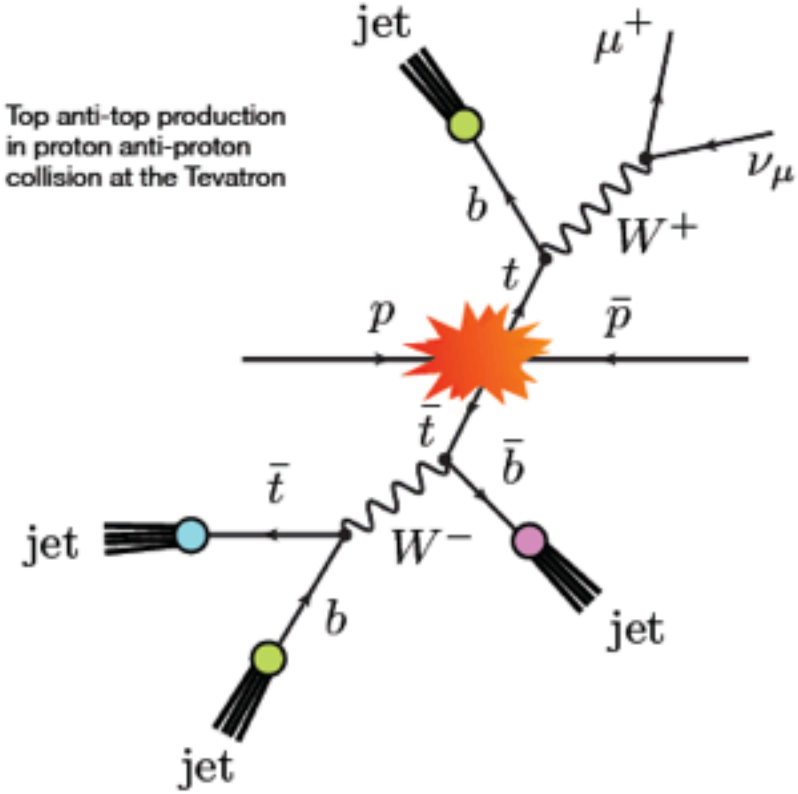
Ledermann et al.

L.L. coined also the term "God particle"

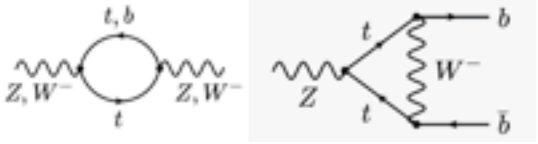
Nobel Prize in 1988 for muon neutrino

# QCD matter sector: top



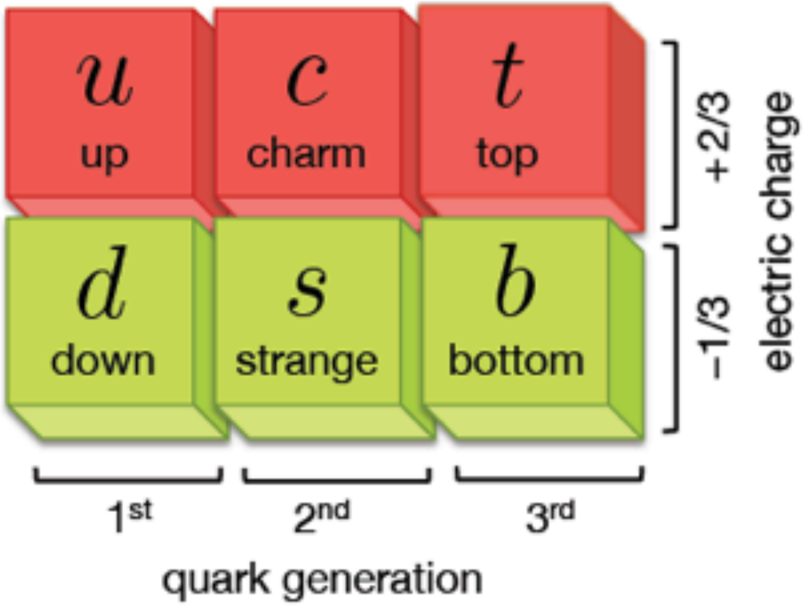

Top anti-top production in proton anti-proton collision at the Tevatron

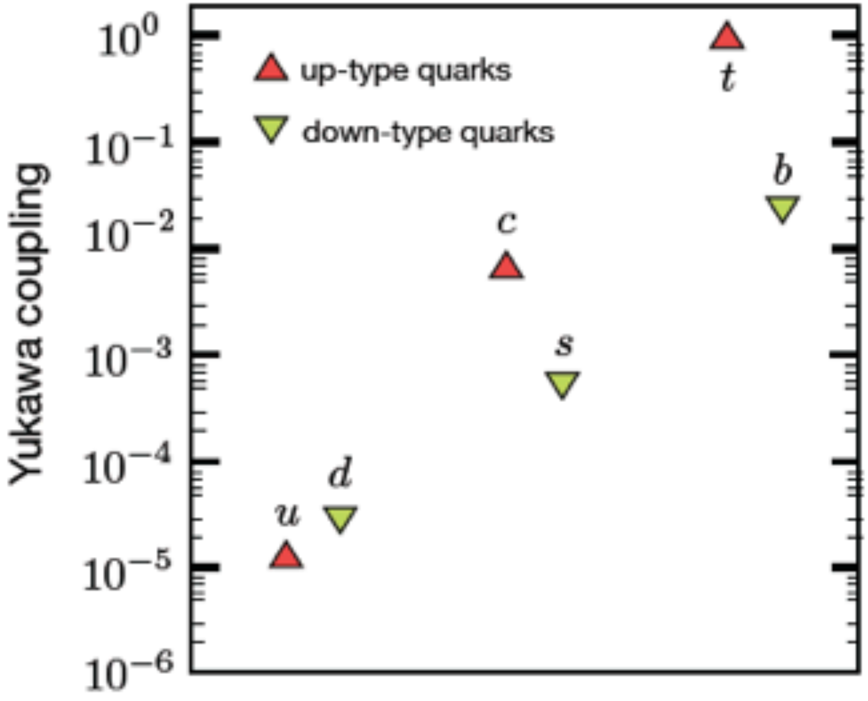
by around 1994 electroweak precision fits point towards mass in range 145-185 GeV
(vector boson mass and couplings are sensitive to top mass)



eventually discovered in 1995 by CDF and DØ at FNAL
(mass nowadays know to about 1 GeV)

# QCD matter sector: 3 generations

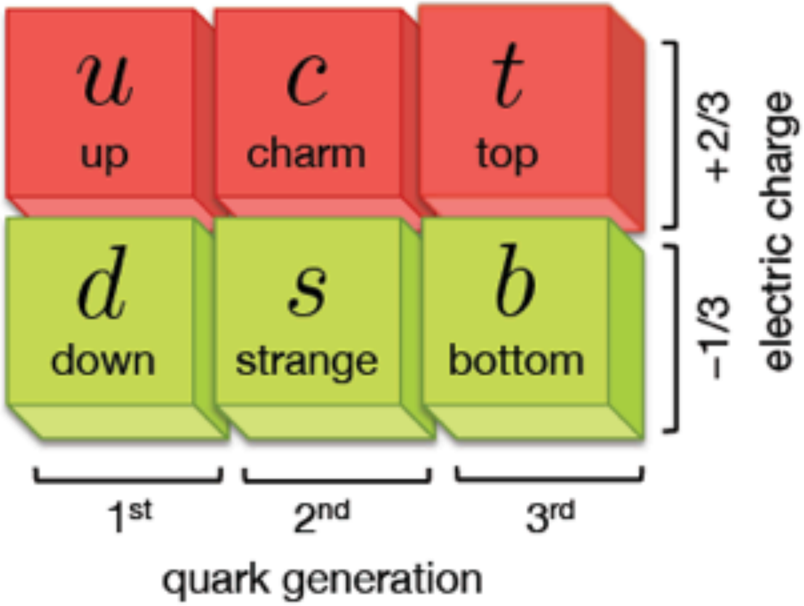


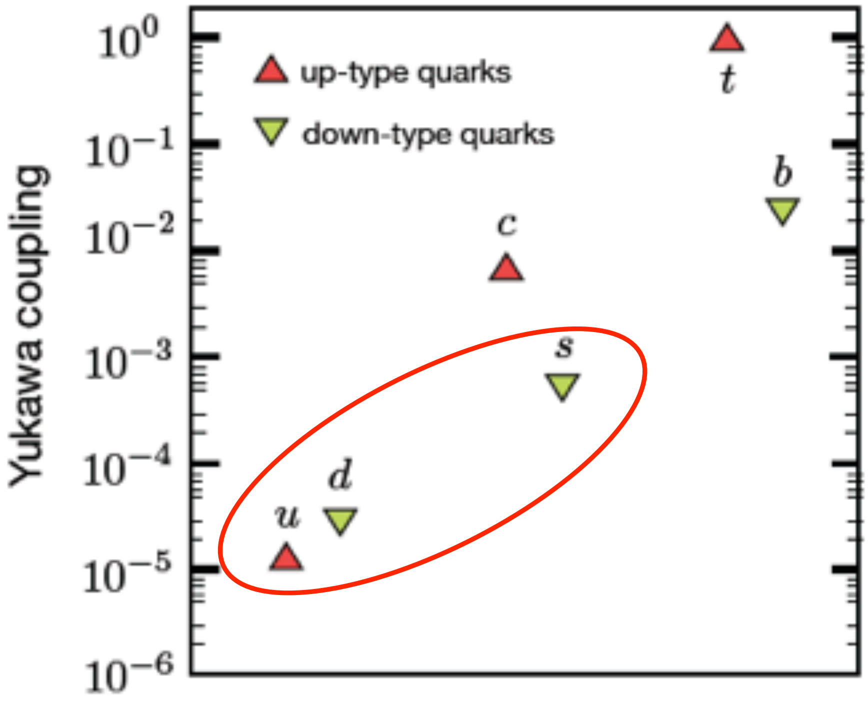


- masses of six quarks range from O(MeV) to about 175 GeV

  why the masses are split by almost six orders of magnitude remains a big mystery

# QCD matter sector: 3 generations





- masses of six quarks range from O(MeV) to about 175 GeV

  why the masses are split by almost six orders of magnitude remains a big mystery

- masses of u, d, s quarks are lighter than 1 GeV (proton mass)

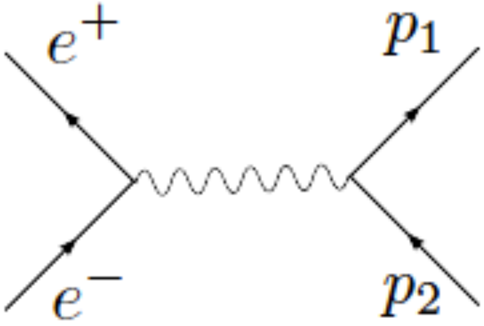
  in the limit of vanishing u,d,s masses there is an exact $SU(3)_{flavor}$ symmetry

# further evidence for color quantum number

- **color can be probed directly in $e^+e^-$ collisions**

  **idea:**

  production of fermion pairs (leptons or quarks) through a virtual photon sensitive to electric charge and number of degrees of freedom



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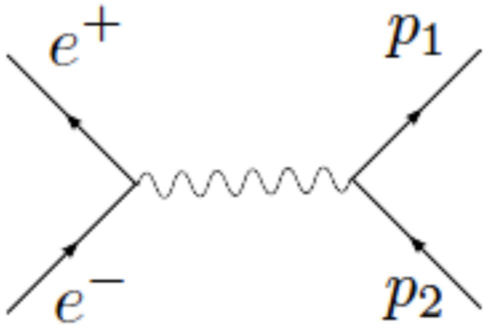

- hence, investigate quarks through "**R ratio**"

$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$

assumed number
of colors of quark

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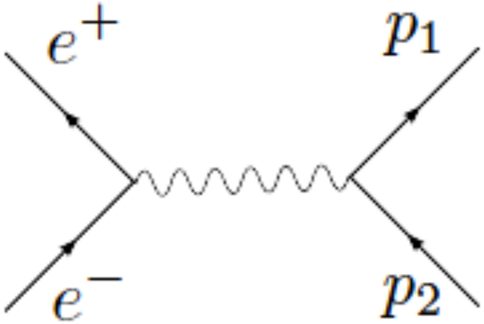

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$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$

electric charge of quark [in units of e]

assumed number of colors of quark

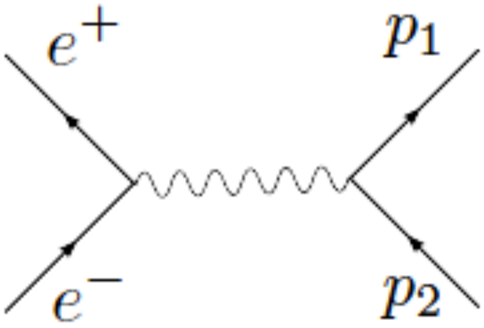
  - in LO described by process $\mathbf{e^+e^- \to q\bar{q}}$

# further evidence for color quantum number

- **color can be probed directly in $e^+e^-$ collisions**

  **idea:**

  production of fermion pairs (leptons or quarks) through a virtual photon sensitive to electric charge and number of degrees of freedom



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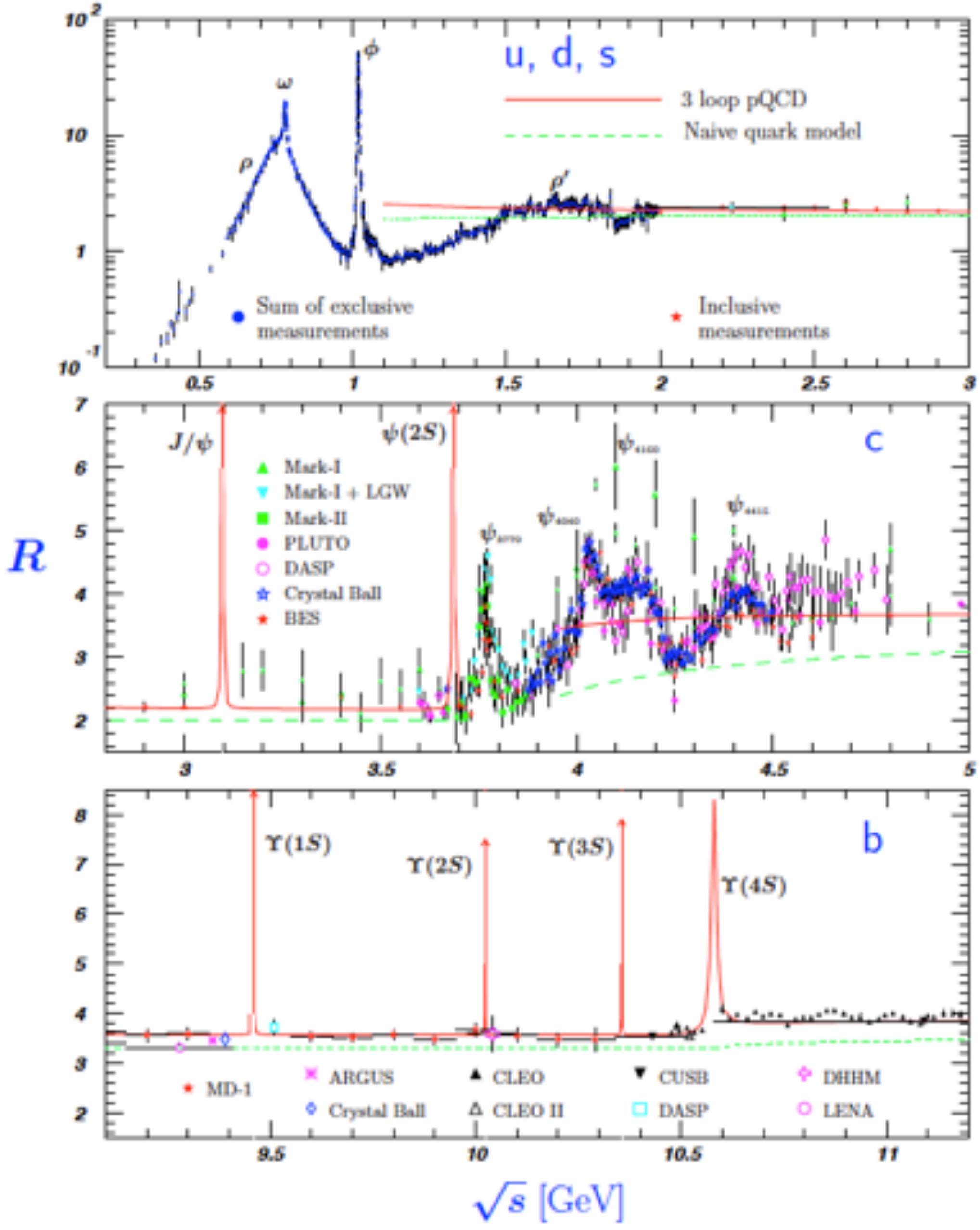
sum over active quarks

- in LO described by process $\mathbf{e^+e^- \rightarrow q\bar{q}}$
- each active quark is produced in one out of $N_C$ colors above kinematic threshold

# experimental results for R ratio



$$
\begin{aligned}
R_{u,d,s} &= 3 \times \left[ \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 \right] \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
R_{u,d,s,c} &= R_{u,d,s} + 3 \times \left(\frac{2}{3}\right)^2 \\
&= \frac{10}{3}
\end{aligned}
$$

$$
\begin{aligned}
R_{u,d,s,c,b} &= R_{u,d,s,c} + 3 \times \left(-\frac{1}{3}\right)^2 \\
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$$

**caveats:**

- higher order corrections
- mass effects near threshold

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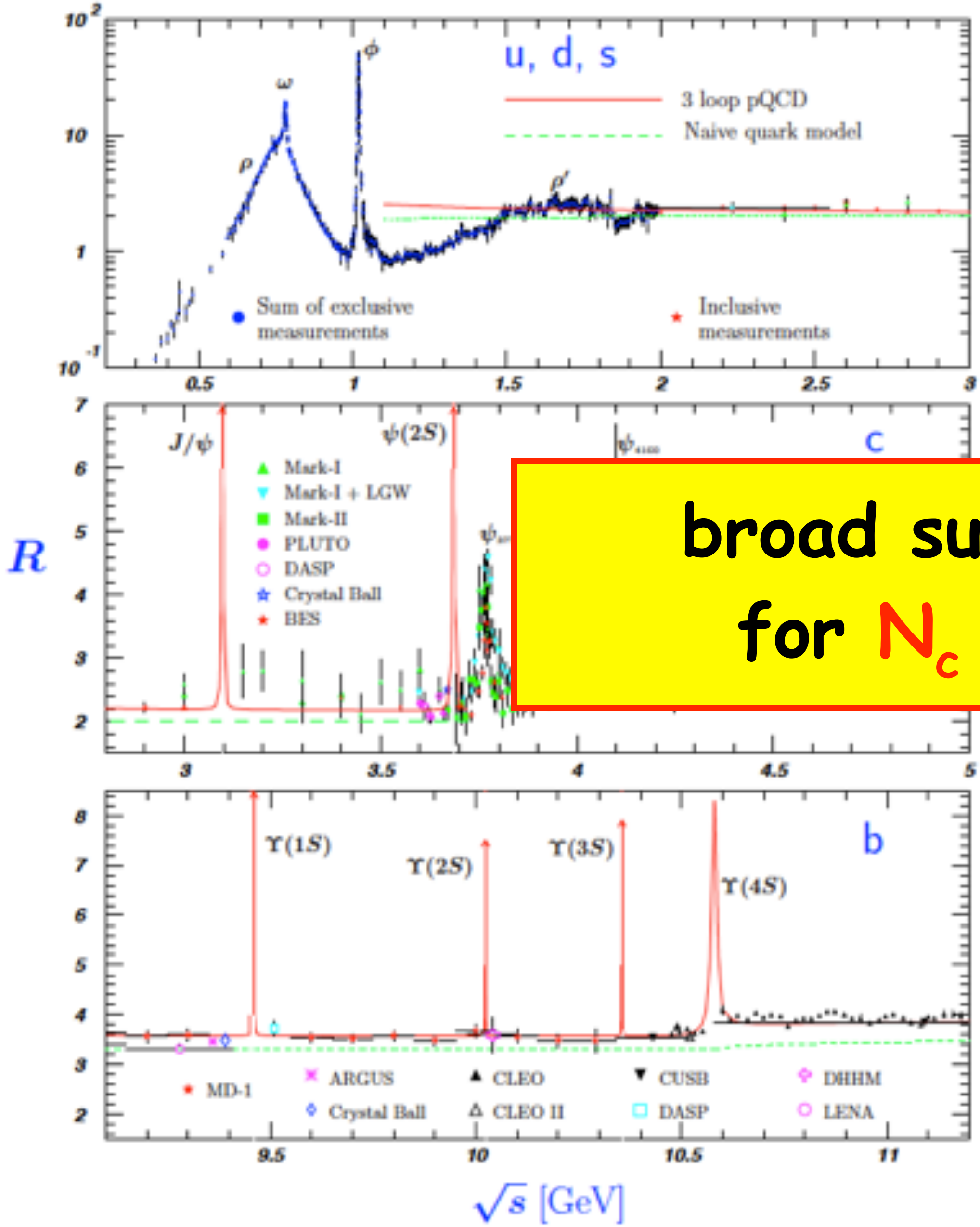

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**broad support**
**for $N_c = 3$**

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# QCD color interactions heuristically

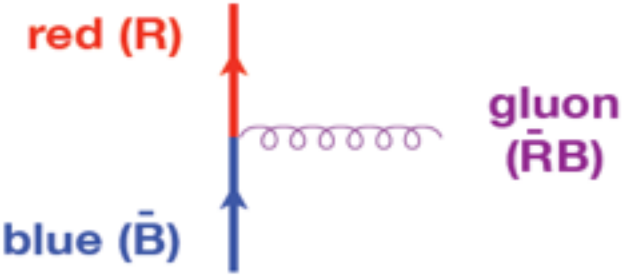
- QCD color quantum number is mediated by the **gluon**

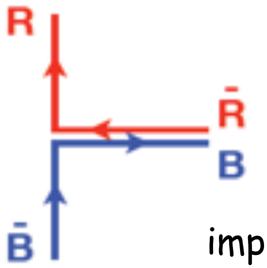
  analogous to the photon in QED

- gluons are changing quarks from one color to another

  as such they must also carry a color charge (unlike the charge neutral photon in QED)

**example:**





**"color flow"**

important calculational tool

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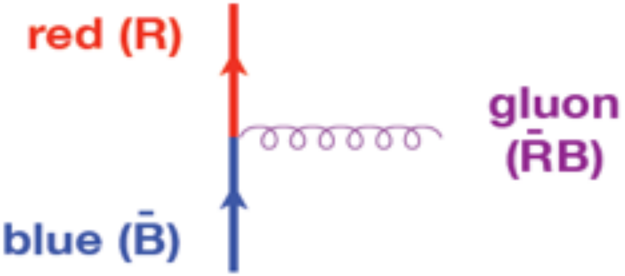

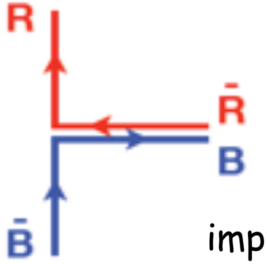

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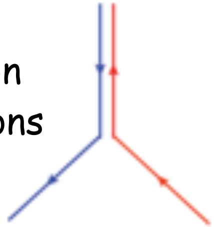
- color charge of each gluon represented by a 3x3 matrix in color space
  conventional choice: express $t^a$ (a=1…8) in terms of **Gell-Mann matrices**

typical color interaction between quarks and gluons



$$\underset{\bar{\psi}_i}{(1,0,0)} \quad \underset{t^1_{ij}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{\psi_j}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

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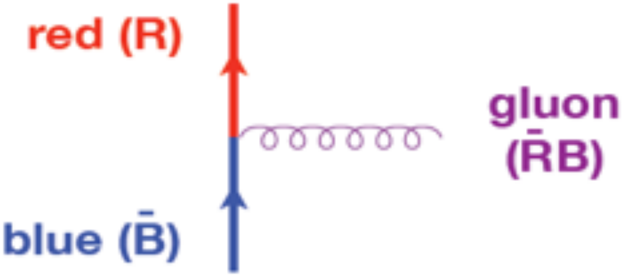

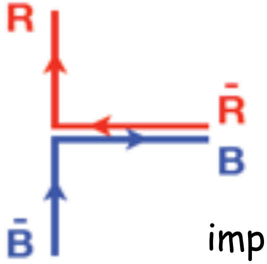

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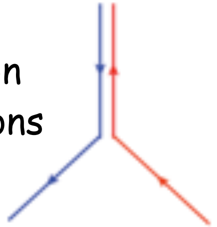

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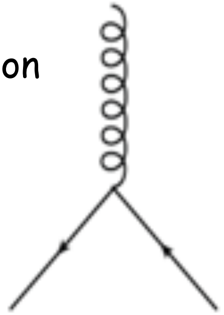
more formal expression
as **Feynman rule**
[only color structure here]



$$\bar{\psi}_i\, t^A_{ij}\, \psi_j$$

# QCD: an unbroken SU(3) Quantum Field Theory

**guiding principle for all field theories**: **local gauge invariance** of the underlying Lagrangian

i.e., redefining the quark and gluon fields independently at each space-time point has no impact on the physics

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spin-$\frac{1}{2}$ quark fields
come as colors triplets
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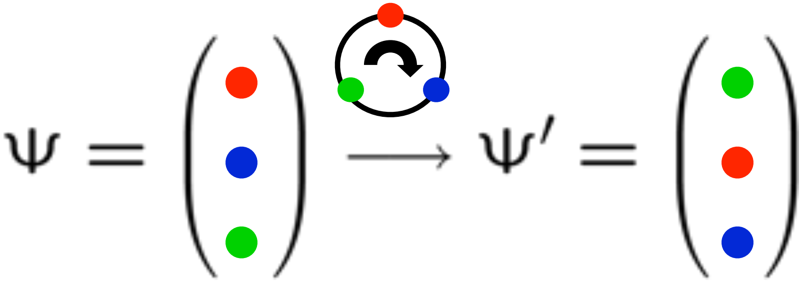

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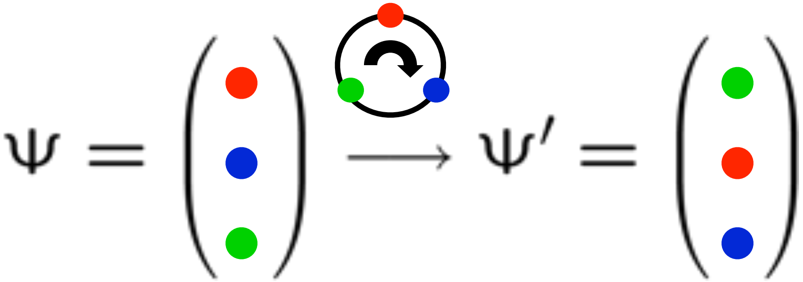

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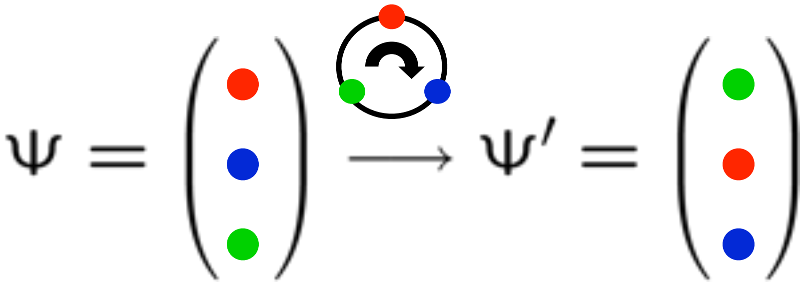

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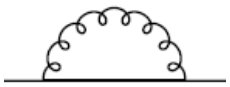
non-Abelian **group structure**:

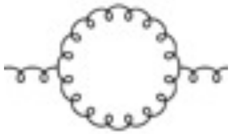
- Lie algebra: $[T_a, T_b] = i\, f_{abc} T_c$

- invariants ("color factors") :

$T_F = 1/2$



$C_F = 4/3$



$C_A = 3$

# experimental support for SU(3)

- **color factors are not just math**

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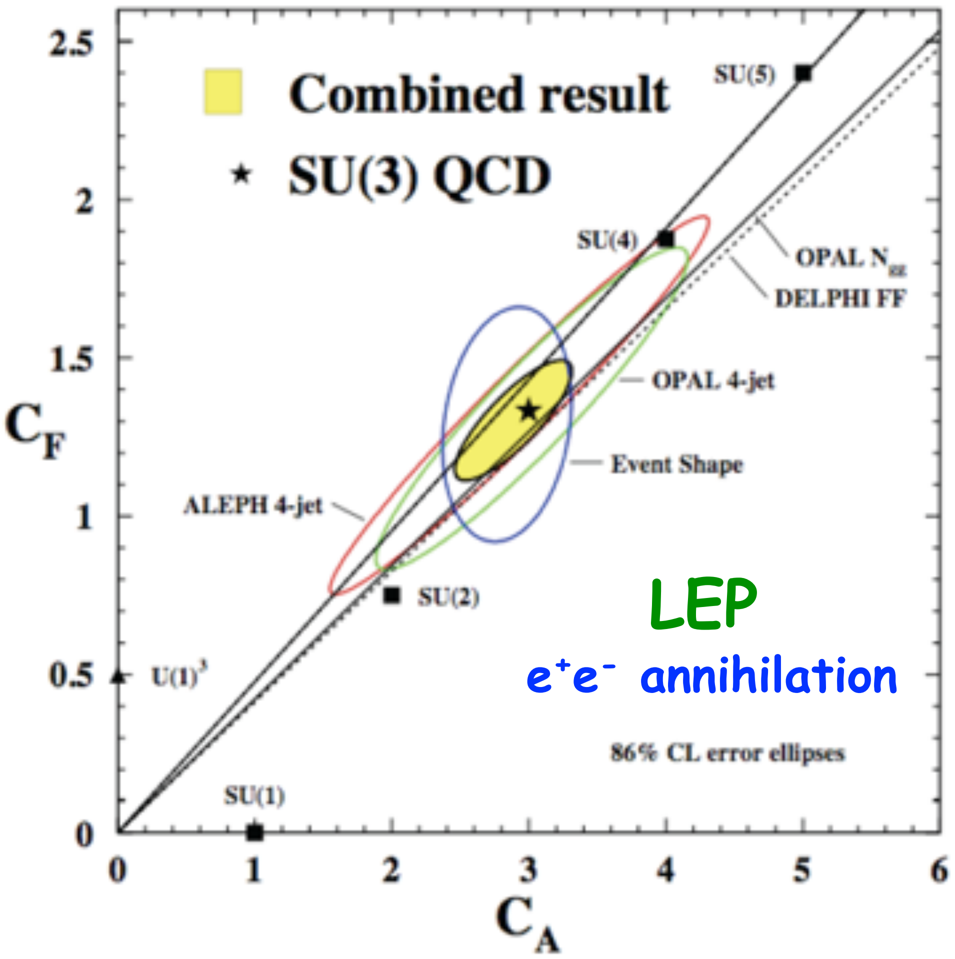
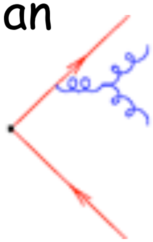


- angular correlations between four jets depend on $C_A/C_F$ and $T_F/C_F$
- sensitivity to non-Abelian three-gluon-vertex

LO: Ellis, Ross, Terrano



# QCD Lagrangian & Feynman rules

$L_{QCD}$ encodes all physics related to strong interactions

for perturbative calculations we simply read off the **Feynman rules**

$$\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} \;=\;& \bar{\Psi}(i\partial_\mu\gamma^\mu - m)\Psi \\
-\;& (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\
-\;& g\bar{\Psi}A^a_\mu T_a\gamma^\mu\Psi \\
-\;& \frac{1}{2}g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)f_{abc}A^{\mu b}A^{\nu c} \\
-\;& \frac{1}{4}g^2 f_{abc}A^b_\mu A^c_\nu f_{ade}A^{\mu d}A^{\nu e}
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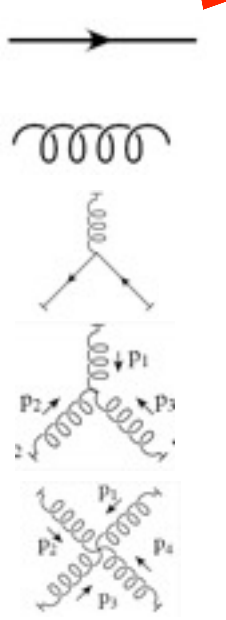

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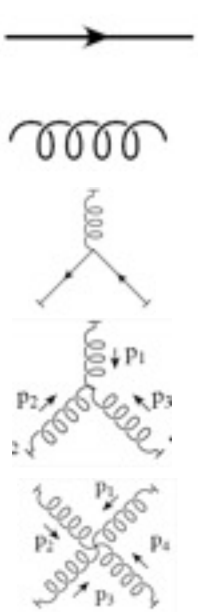

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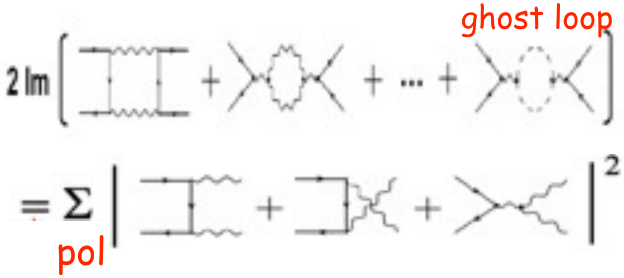
technical complications due to the gauge-fixing & ghost terms:

**gauge-fixing:** needed to define gluon propagator; breaks gauge-invariance but all physical results are independent of the gauge

**ghosts:** cancel unphysical degrees of freedom → unitarity



# recall: gauge invariance in QED

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\mathcal{L}_{\mathbf{QED}} &= \mathcal{L}_{\mathbf{Dirac}} + \mathcal{L}_{\mathbf{Maxwell}} + \mathcal{L}_{\mathbf{int}} \\
&= \bar{\Psi}(\mathrm{i}\not{\partial} - \mathrm{m})\Psi - \frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - \mathrm{q}\bar{\Psi}\gamma_\mu\Psi\mathbf{A}^\mu \\
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electromagnetic vector potential $\mathbf{A}_\mu$

field strength tensor $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu$

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invariant under **local gauge (phase) transformation**

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- dictates interaction term
- photon mass term would violate gauge invariance
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electromagnetic vector potential $A_\mu$ photon field carries no electric charge

field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ field strength itself gauge invariant

covariant derivative $D_\mu = \partial_\mu + iq\,A_\mu$ "covariant" = $D_\mu\psi$ transforms as $\psi$

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electromagnetic ve

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**more cumbersome to demonstrate for QCD**

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# one more look at the QCD Lagrangian

- Yang and Mills proposed in 1954 that the local "phase rotation" in QED could be generalized to non Abelian groups such as SU(3)

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a = 1,…,8

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8 generators

- QCD interaction is flavor blind

- **coupling $g_s$ is the only parameter** (masses have e-w origin)

# take home message for part I
## the foundations

- **QCD is based on a simple Lagrangian but has a rich phenomenology**

- **QCD is based on the non Abelian gauge group SU(3)**
  - number of colors and group structure can be tested experimentally
  - concept of local gauge invariance dictates interactions
  - similarities to QED, yet profound differences (and more to come)
  - color leads to self-interactions between “force carrying” gluons
  - perturbation theory can be based on a short list of Feynman rules

- **color algebra decouples and can be performed separately**
  - color factors can be expressed in terms of two Casimirs: $C_A$ and $C_F$

# Part II

## the QCD toolbox

asymptotic freedom, IR safety,
QCD final state, factorization

# dichotomy of QCD

the gauge principle is elegant and powerful but any theory must ultimately stand (or fall) by its success (or failure)

QCD is the theory of **strong** interactions

– how can we make use of **perturbative** methods?

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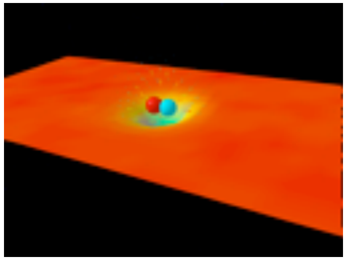
**confinement**



D. Leinweber

non-perturbative
structure of hadrons

e.g. through **lattice QCD**

**asymptotic freedom**

hard scattering
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the gauge principle is elegant and powerful but any theory must ultimately stand (or fall) by its success (or failure)

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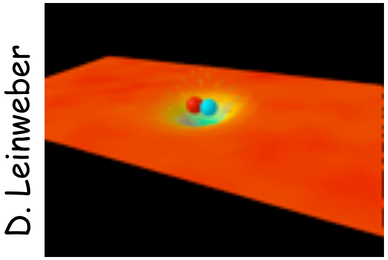

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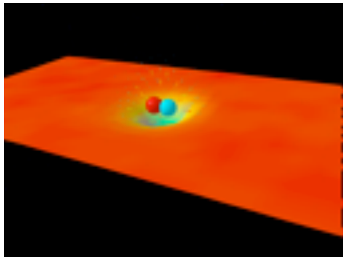

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interplay

probing hadronic structure with
weakly interacting quanta of asymptotic freedom

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**Gross, Wilczek;**
**Politzer ('73/'74)**
**Nobel prize 2004**

value of strong coupling $\alpha_s = g^2/4\pi$ depends on distance r (i.e., on energy Q)

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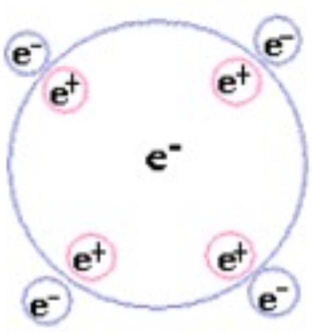

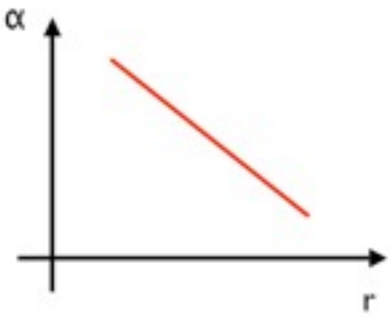

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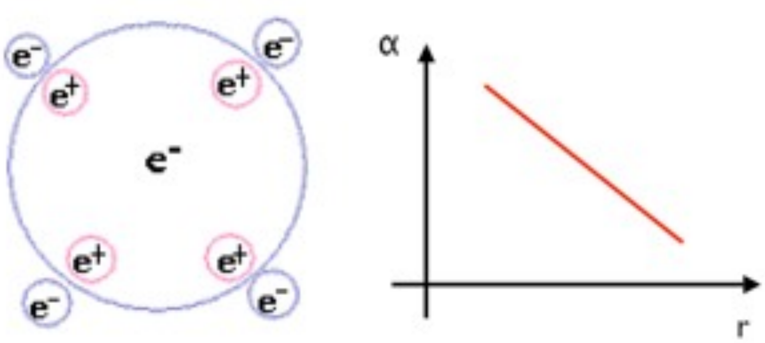

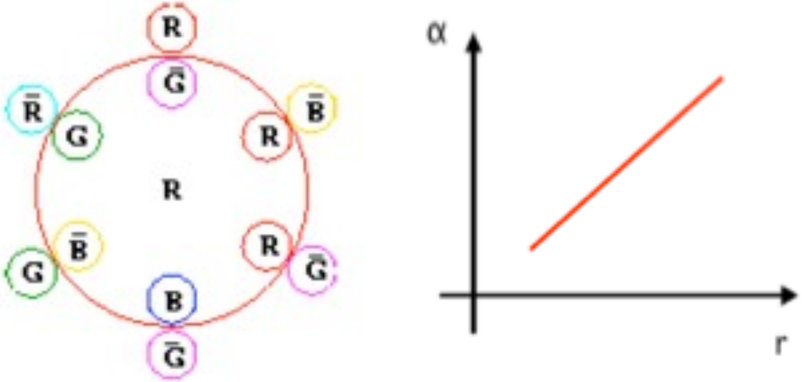

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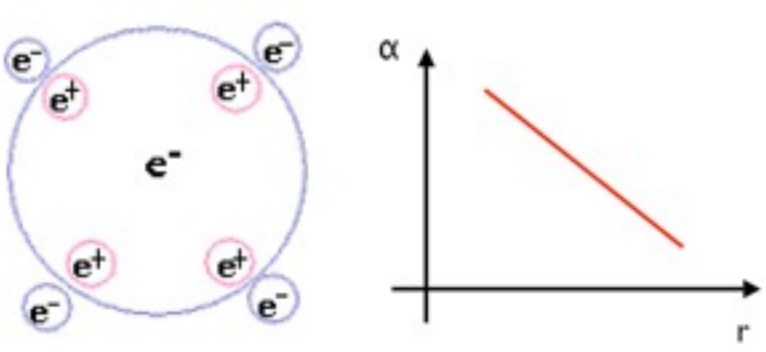

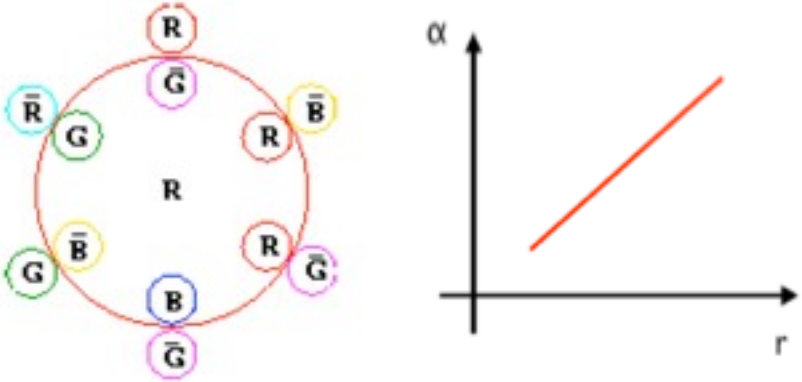

who wins ?

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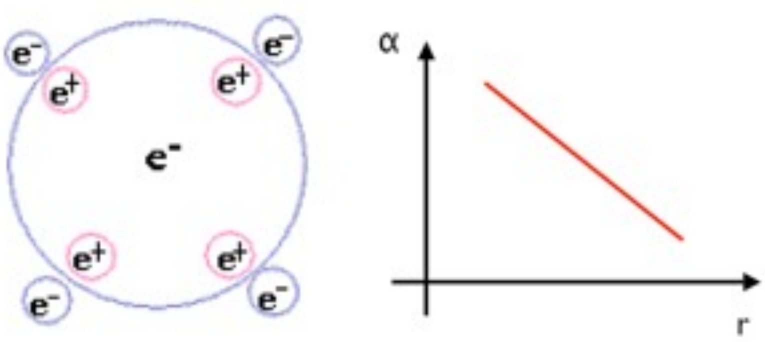

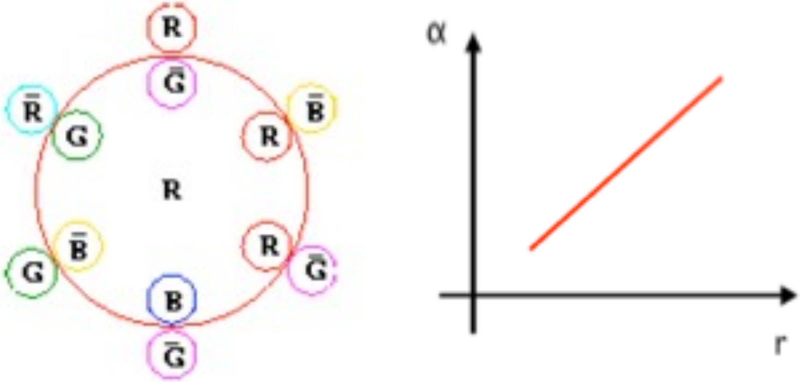

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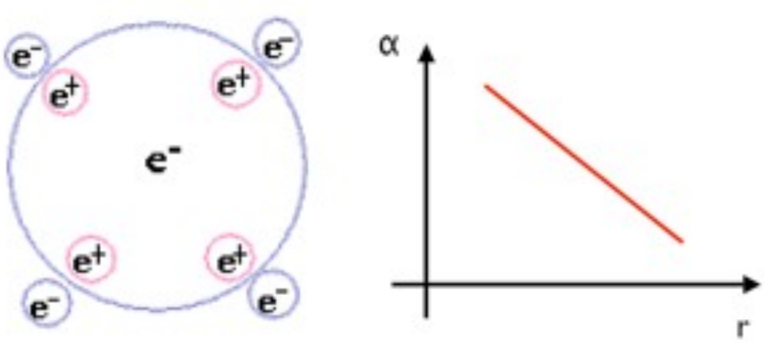

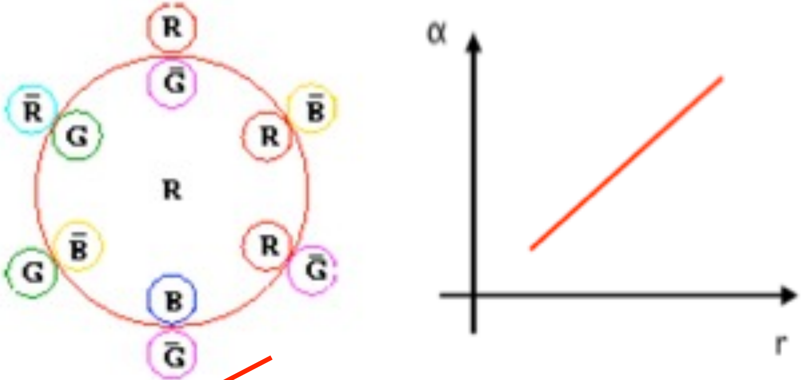

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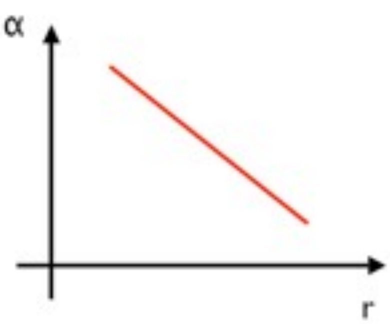

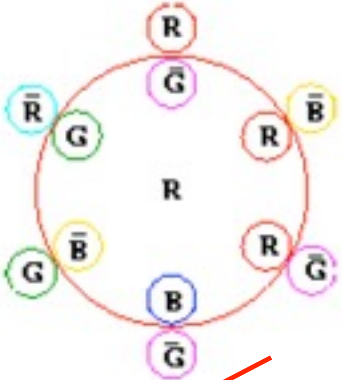

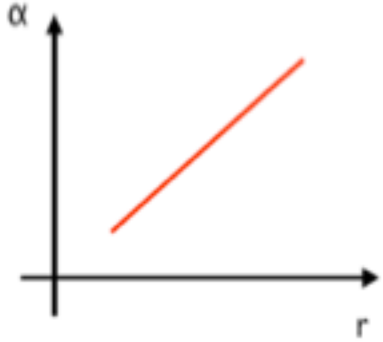

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typical hadronic scale O(200 MeV)
$\Lambda$ depends on $N_f$, pert. order and scheme

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van Ritbergen, Vermaseren, Larin

('71), '73 '74 '80 '97

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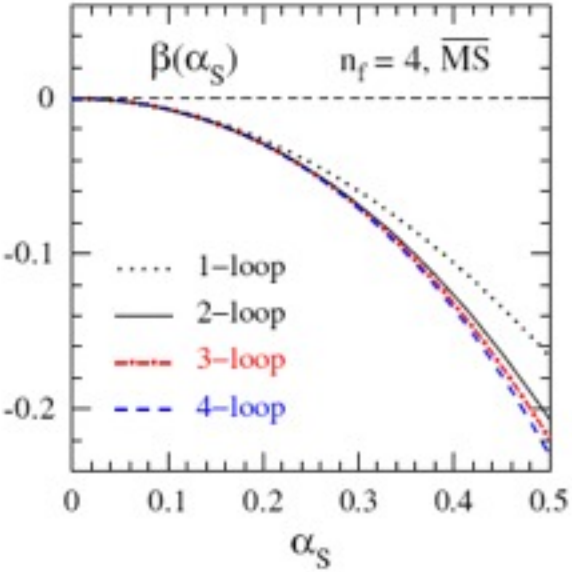

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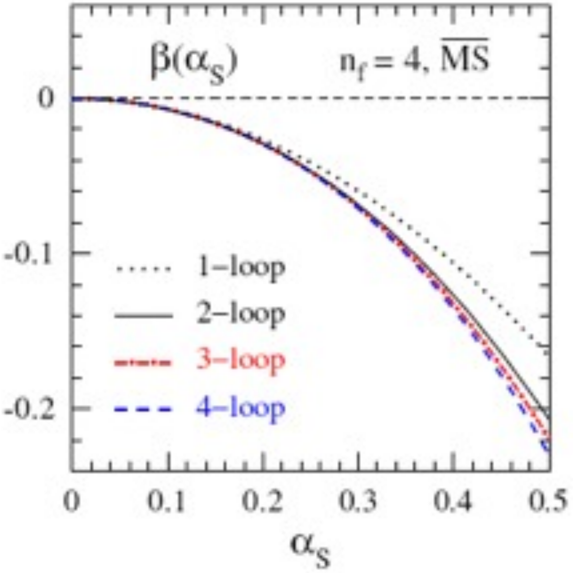

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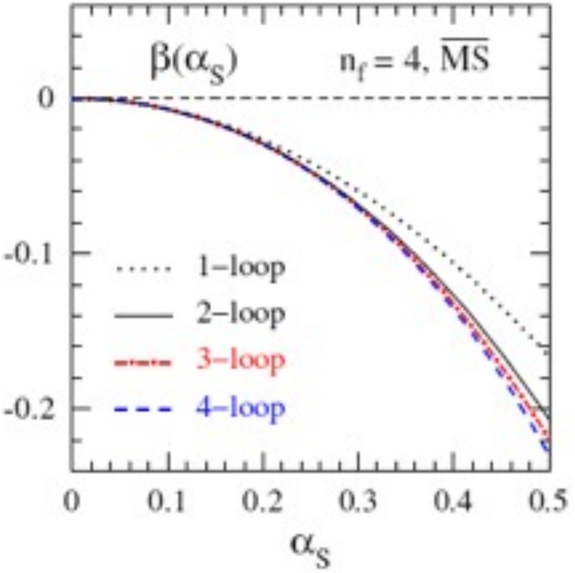

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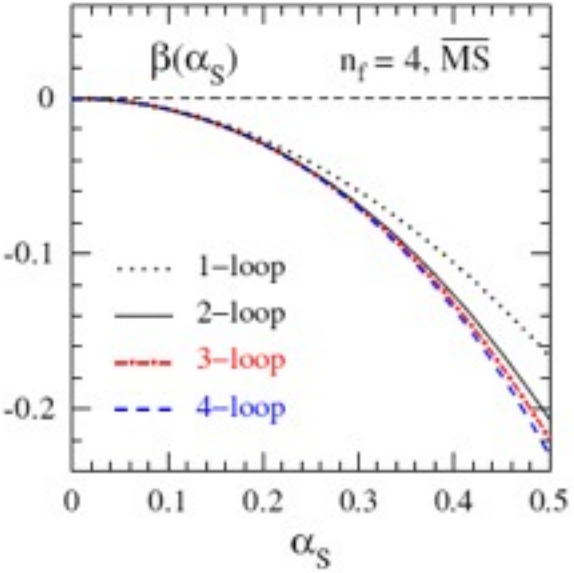

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**tells us how $\alpha_s$ varies with scale but not its absolute value at $\mu_0$**

$a_s(\Lambda^2) = \infty$

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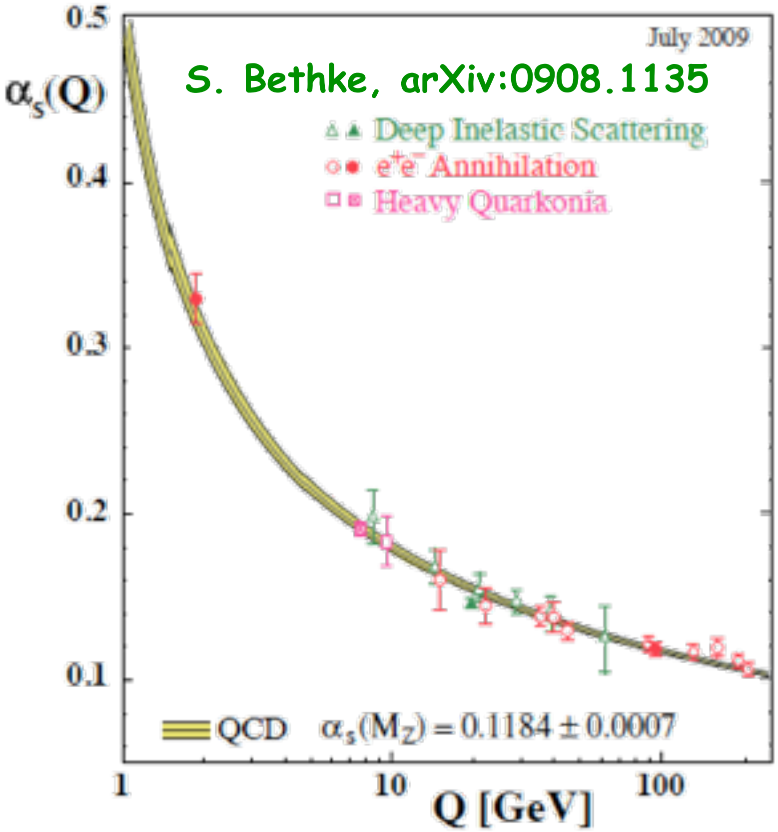
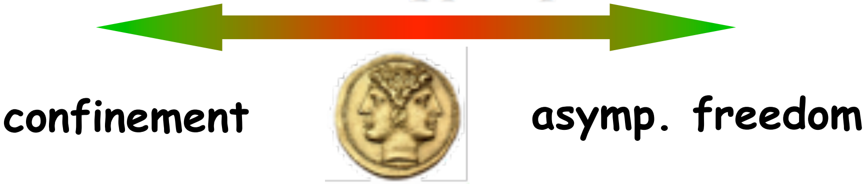
**1st example of a renormalization group equation**

# consistent picture from many observables



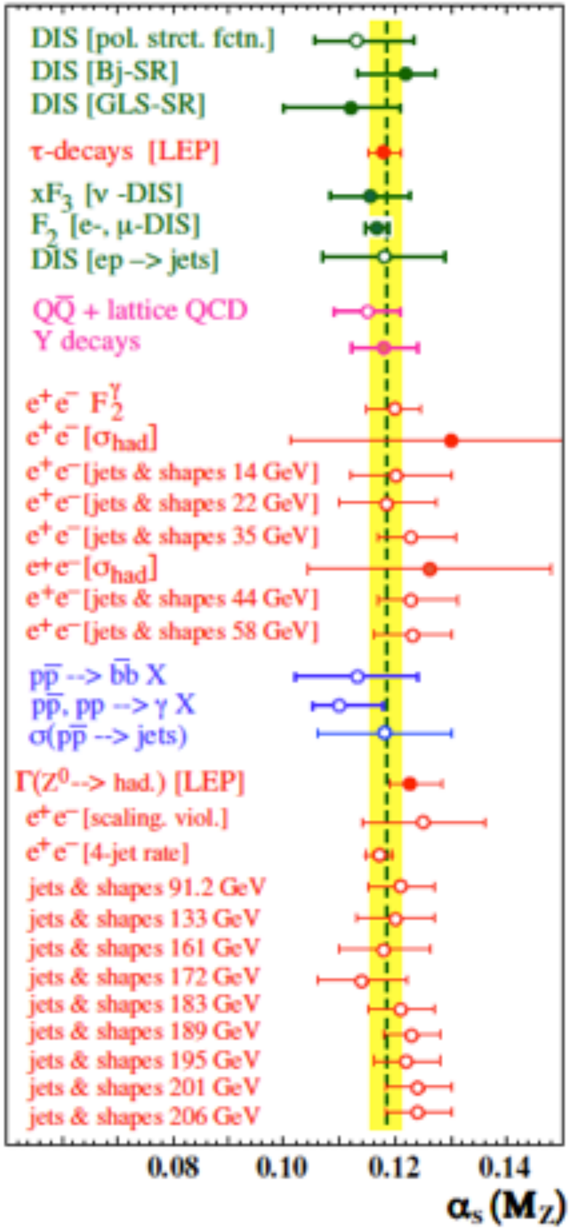
S. Bethke, arXiv:0908.1135



confinement

asymp. freedom

**exp. evidence for $\log(Q^2)$ fall-off is persuasive**



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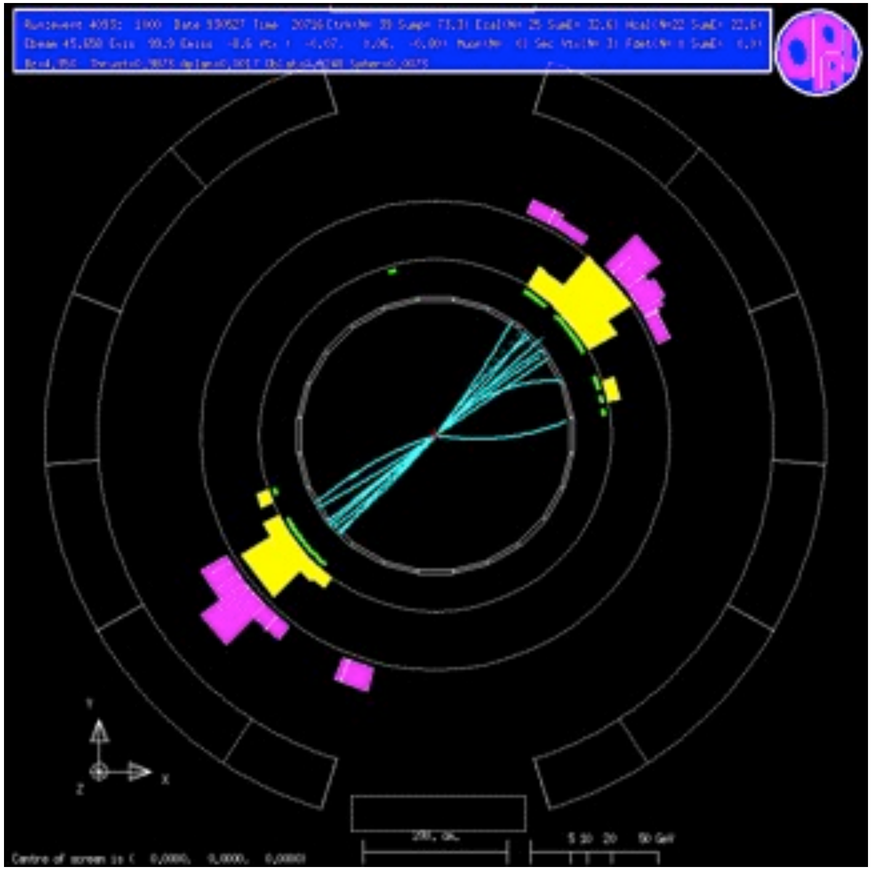
let's study electron-positron annihilation to see what this is all about ...

# $e^+e^-$ annihilation: the QCD guinea pig

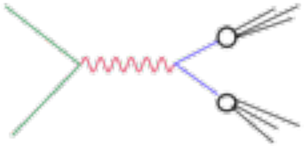
1989-2000

most of the hadronic events at CERN-LEP had **two back-to-back jets**

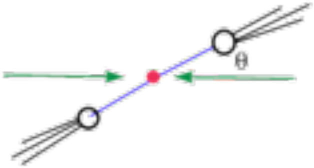


**jet**: pencil-like collection of hadrons

- jets resemble features of underlying 2->2 hard process $e^+e^- \to q\bar{q}$



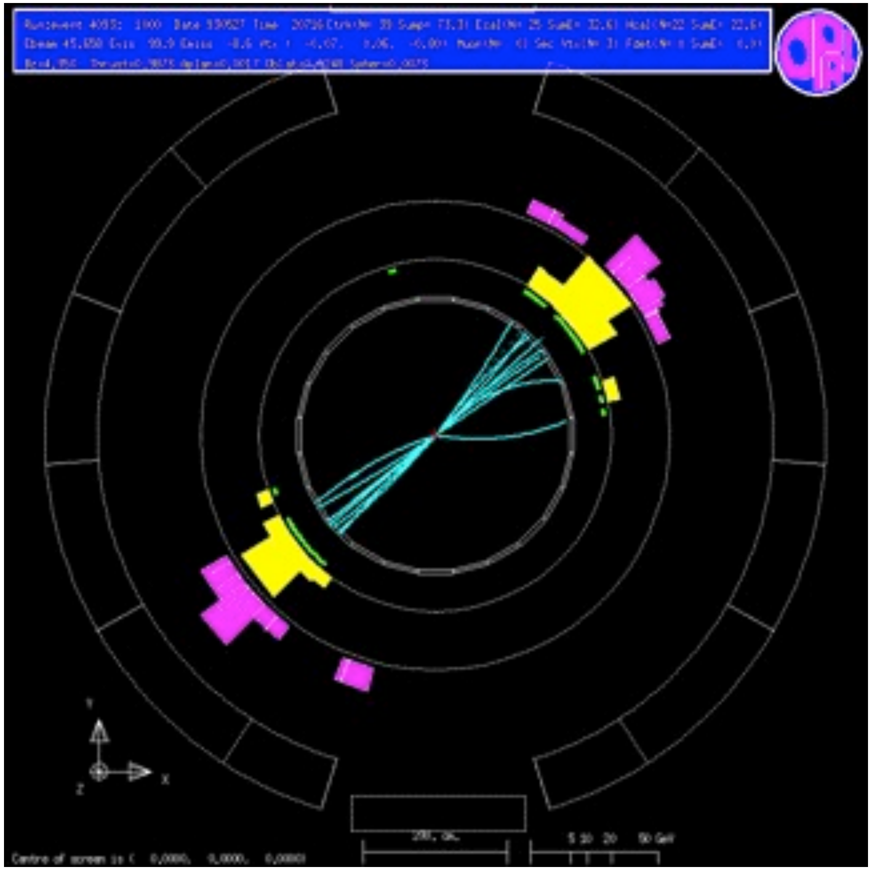
- angular distribution of jet axis w.r.t. beam axis as predicted for **spin-½ quarks**



# $e^+e^-$ annihilation: the QCD guinea pig

1989-2000

most of the hadronic events at CERN-LEP had **two back-to-back jets**



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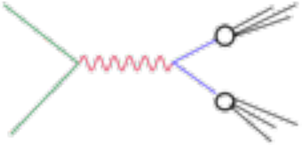
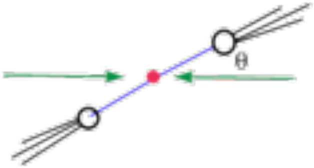


- angular distribution of jet axis w.r.t. beam axis as predicted for **spin-$\frac{1}{2}$ quarks**

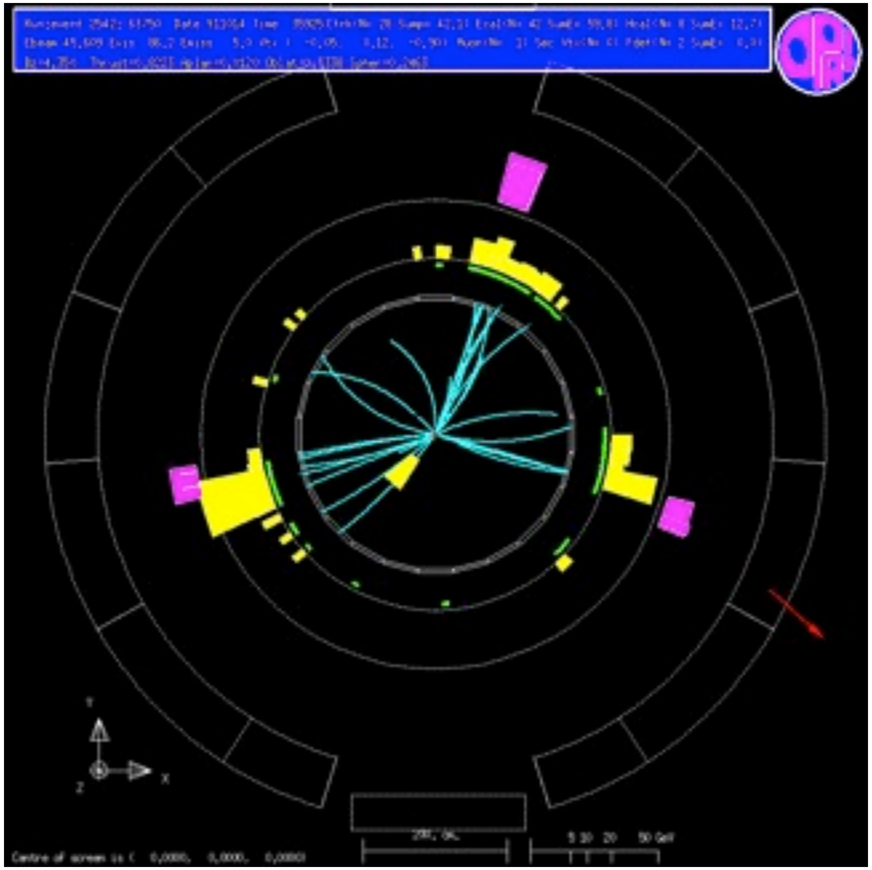


**jets play major role in hadron-hadron collisions at TeVatron, RHIC, LHC**

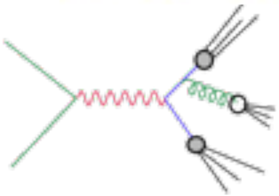
# $e^+e^-$ annihilation: three-jet events

about 10% of the events had a **third jet**

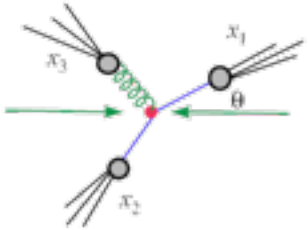
first discovered at
DESY-PETRA in 1979



- jets resemble features of underlying 2->3 hard process $e^+e^- \longrightarrow q\bar{q}g$



- 10% rate consistent with $\alpha_s \simeq 0.1$ (**determination of $\alpha_s$**)
- angular distribution of jets w.r.t. beam axis as expected for **spin-1 gluons**



# recipe for quantitative calculations

(1) identify the final-state of interest and draw all relevant Feynman diagrams

(2) use SU(3) algebra to take care of QCD color factors

(3) compute the rest of the diagram using “Diracology”

traces of gamma matrices, spinors, …

(4) to turn squared matrix elements into a cross section we need to

- account for the available phase space (momentum d.o.f. in final-state)
- integrate out not observed d.o.f.
- normalize by incoming flux

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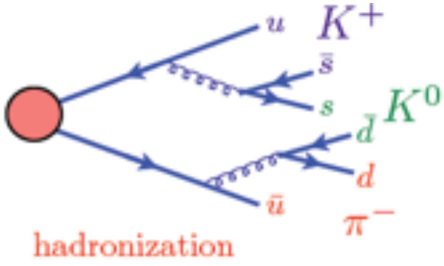

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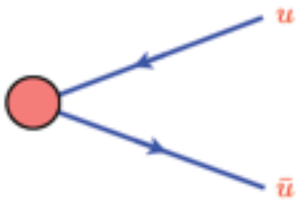
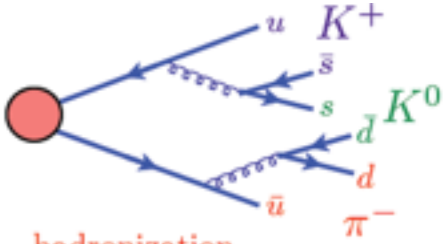
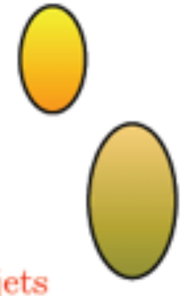


energetic partons



hadronization



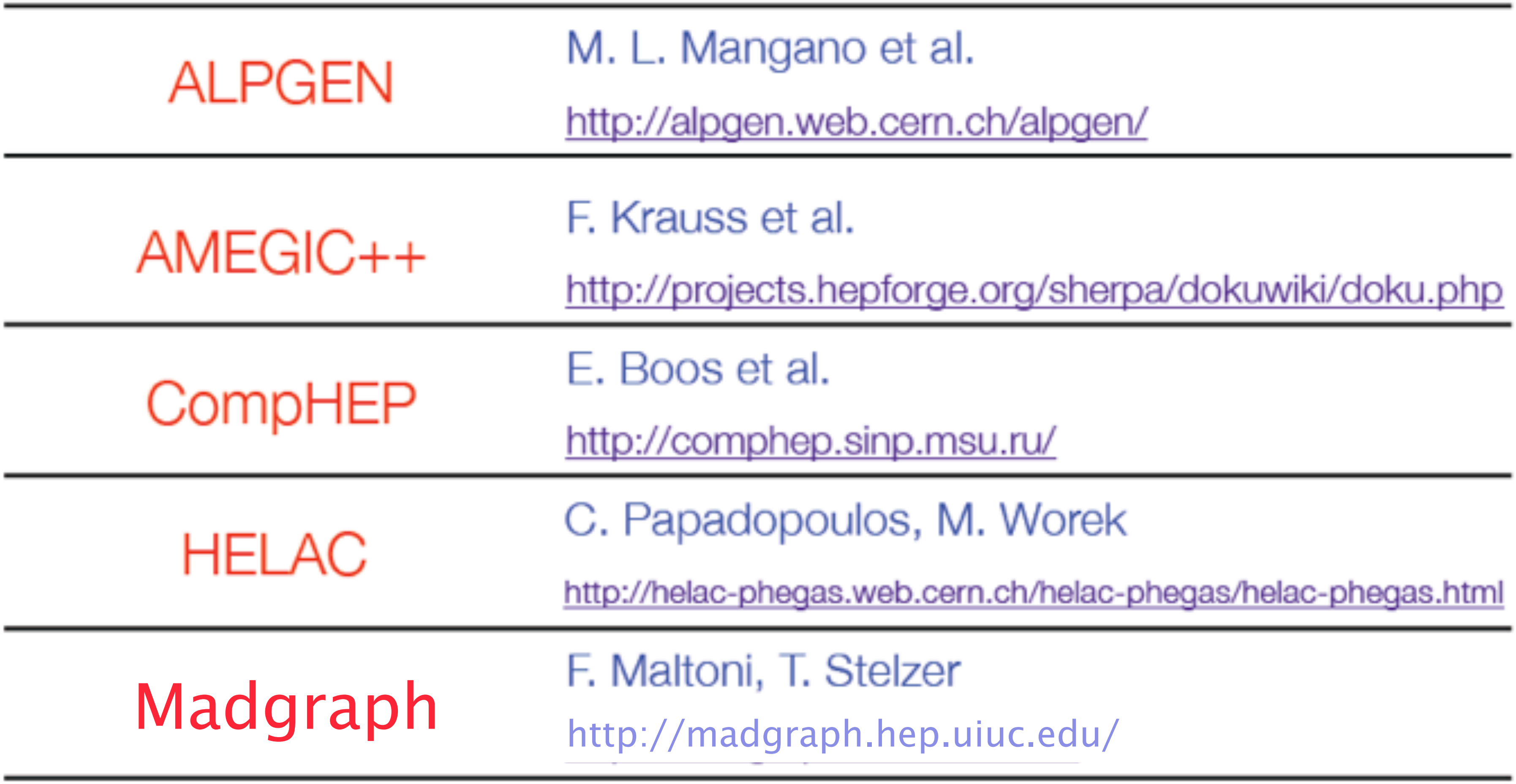
jets

cleanest observables in QCD

will find that most "stuff"
is observed in the directions
of produced quarks & gluons
**parton-hadron duality**

# bunch of automated LO tools

- LO estimates of cross sections are practically a solved problem
- many useful fully automated tools available (limitations for high multiplicities)



| | |
|---|---|
| ALPGEN | M. L. Mangano et al.<br>http://alpgen.web.cern.ch/alpgen/ |
| AMEGIC++ | F. Krauss et al.<br>http://projects.hepforge.org/sherpa/dokuwiki/doku.php |
| CompHEP | E. Boos et al.<br>http://comphep.sinp.msu.ru/ |
| HELAC | C. Papadopoulos, M. Worek<br>http://helac-phegas.web.cern.ch/helac-phegas/helac-phegas.html |
| Madgraph | F. Maltoni, T. Stelzer<br>http://madgraph.hep.uiuc.edu/ |

let's have a closer look at the R-ratio already encountered in Part I

$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$

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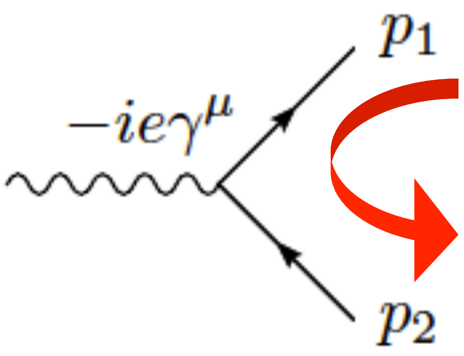
at LO described by:

vertex

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$

spinors for external lines

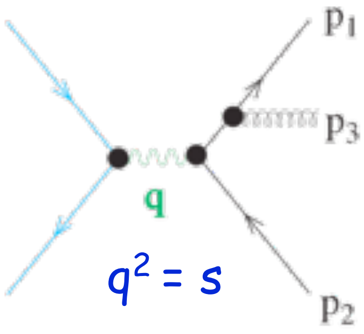


"read against the arrow"

# exploring the QCD final-state: $e^+e^- \rightarrow$ 3 partons

simplest process in pQCD: $e^+e^- \rightarrow q\bar{q}g$

(all partons massless)



$q^2 = s$

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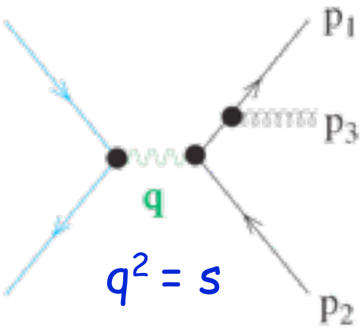

**some kinematics first:**

- energy fractions & conservation: $x_i \equiv \frac{2p_i \cdot q}{s} = \frac{E_i}{\sqrt{s}/2}$ $\sum x_i = \frac{2(\sum p_i) \cdot q}{s} = 2$

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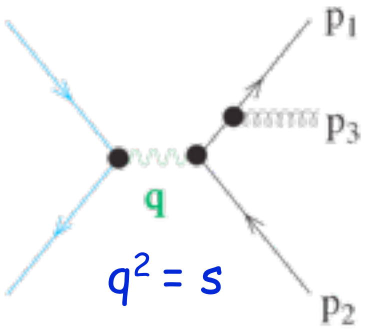

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$$\Leftrightarrow \quad x_1 x_3 (1 - \cos\theta_{13}) = 2(1 - x_2)$$

(other angles by cycl. permutation)

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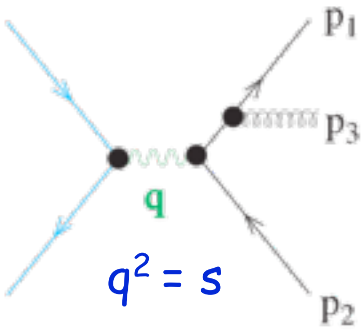

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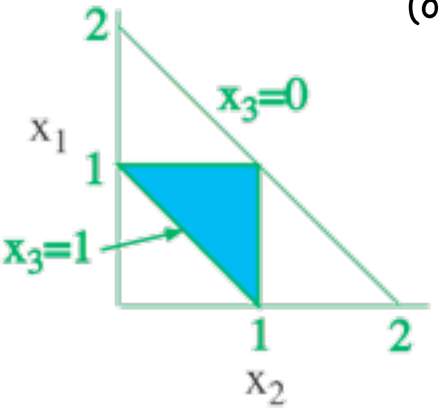
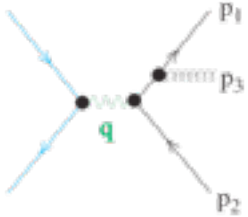
$$\Rightarrow \boxed{0 \leq x_i \leq 1}$$

allowed values for $x_i$ lie within a triangle

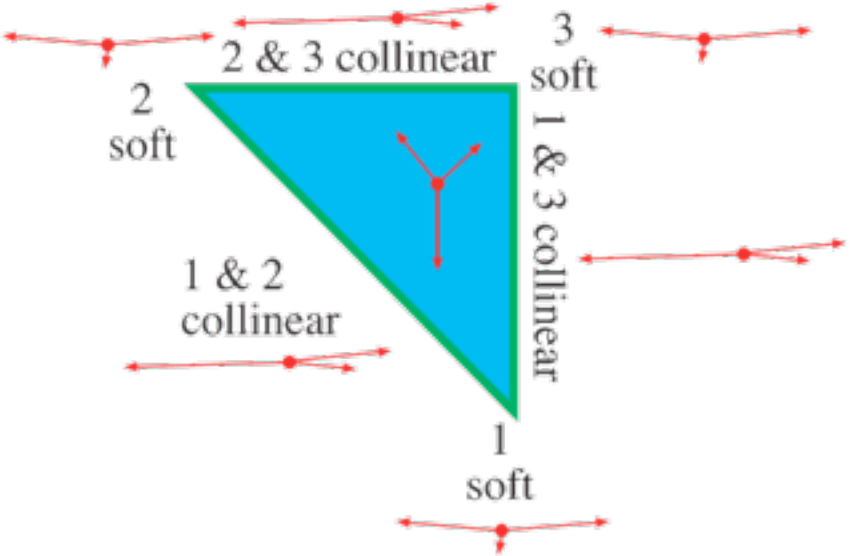


massless "Dalitz plot"

# collinear and soft configurations



at the boundaries of phase space we encounter
**special kinematic configurations**:



- “edges”: **two partons collinear**

$$\boxed{\text{e.g. } \theta_{13} \rightarrow 0 \Leftrightarrow x_2 \rightarrow 1}$$

- “corners”: **one parton soft**

$$\boxed{p_i^\mu \longrightarrow 0 \Leftrightarrow x_i \longrightarrow 0}$$

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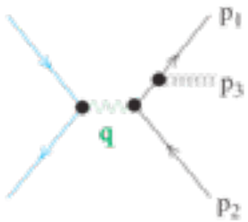

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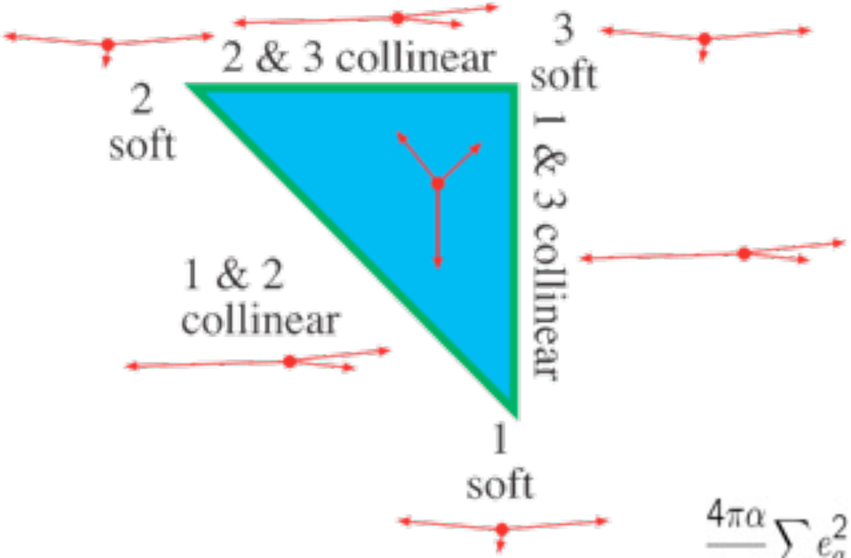

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$$\frac{4\pi\alpha}{s}\sum e_q^2$$

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

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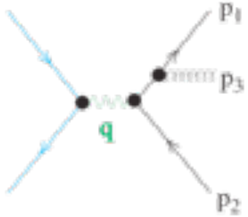

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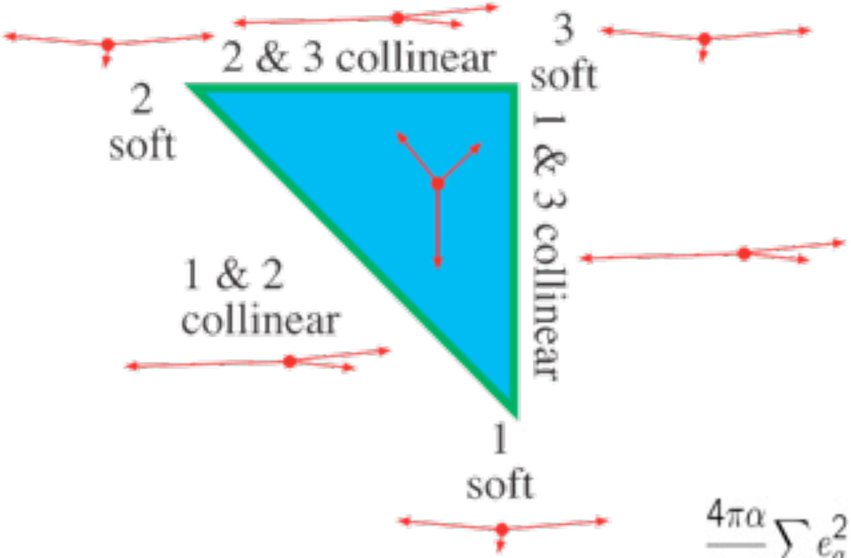

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**collinear singularities:**

$x_1 \to 1$ : gluon ∥ antiquark

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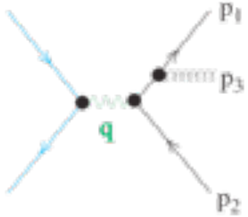

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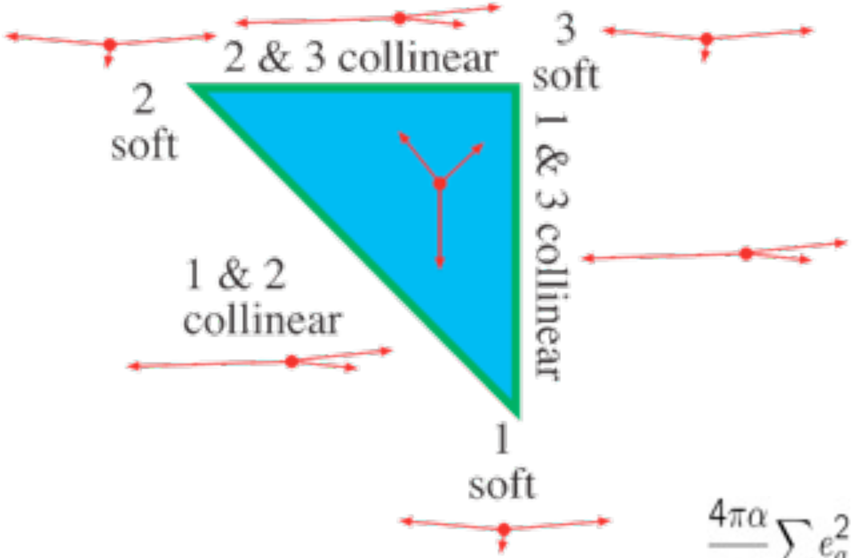

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**soft gluon singularity:**

$x_3 \to 0$ : $p_3 \to 0$

$\leftrightarrow x_1 \to 1$ & $x_2 \to 1$

**collinear singularities:**

$x_1 \to 1$ : gluon $\|$ antiquark

$x_2 \to 1$ : gluon $\|$ quark

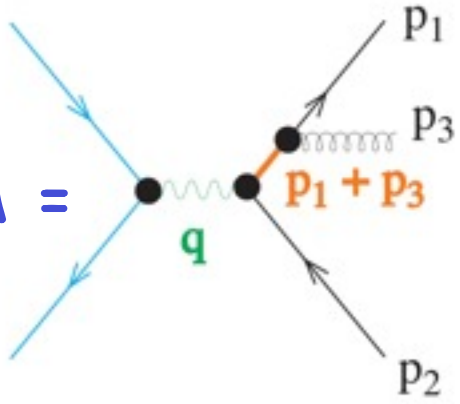
# general nature of these singularities

**soft/collinear limit:**

internal propagator goes on-shell

here: $\frac{1}{(p_1+p_3)^2} = \frac{1}{2E_1E_3(1-\cos\theta_{13})}$

$M =$



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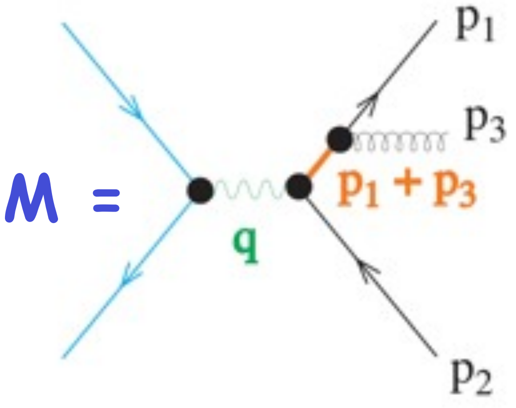

explicit calculation yields:

$$d\sigma \propto \int \underbrace{E_3 dE_3 d\theta_{13}^2}_{\text{phase space factor}} \underbrace{\left[\frac{\theta_{13}}{E_3\theta_{13}^2}\right]^2}_{\text{from } |M|^2} = \int \frac{dE_3}{E_3}\frac{d\theta_{13}^2}{\theta_{13}^2}$$

logarithmically divergent

note: "soft quarks" (here $E_1 \to 0$) never lead to singularities (canceled by numerator)

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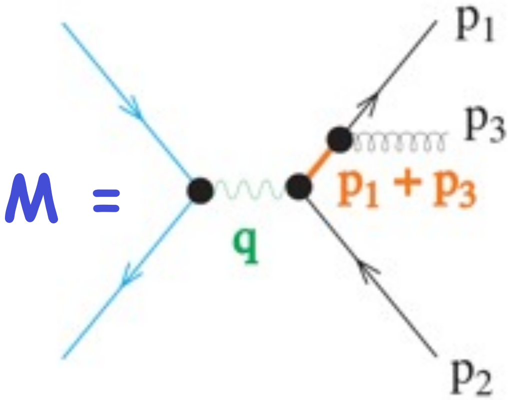

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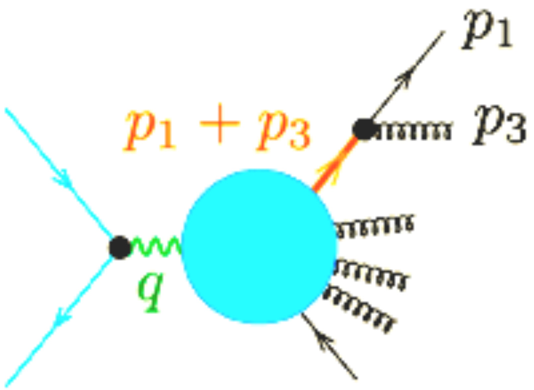
**this structure is generic for QCD tree graphs:**

$$\mathcal{M}_{n+1} \sim [\mathcal{M}_n]\Big|_{1,3\,\text{on-shell}} \frac{\text{spinors}}{(p1+p3)^2}$$



basis for parton-shower MC codes
like PYTHIA, HERWIG, SHERPA, …

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**the lesson is:**

whenever the 2->(n+1) kinematics collapses to an effective 2->n parton kinematics due to

- the emission of a soft gluon
- a collinear splitting of a parton into two partons

we have to be much more careful and work a bit harder!

**this applies to all pQCD calculations**

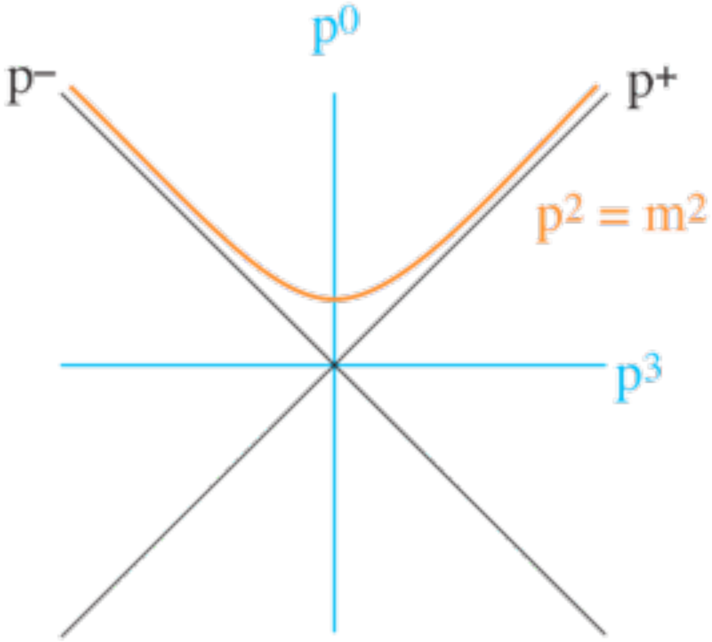
# towards a space-time picture of the singularities

**interlude:** light-cone coordinates

$$p^{\pm} \equiv (p^0 \pm p^3)/\sqrt{2}$$

$$p^2 = 2p^+p^- - \vec{p}_T^{\,2}$$

$$p^- = (p_T^2 + m^2)/2p^+$$



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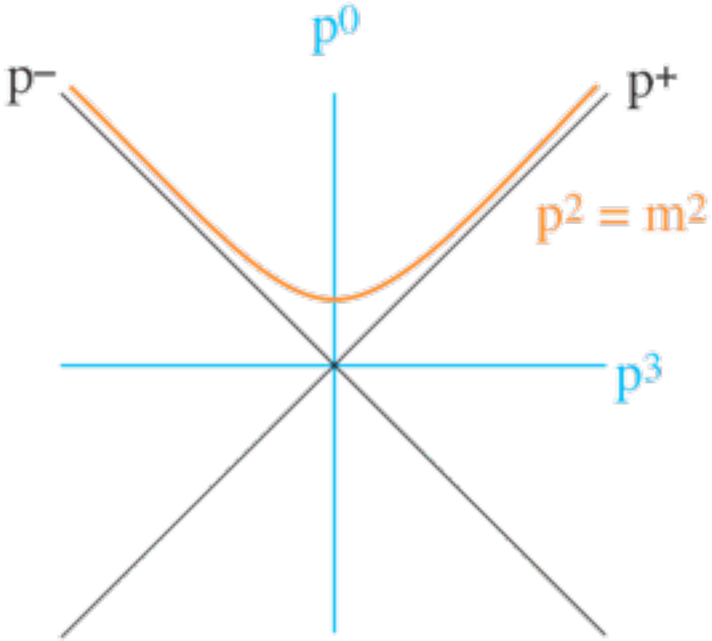

➡ particle with large momentum in $+p^3$ direction has large $p^+$ and small $p^-$

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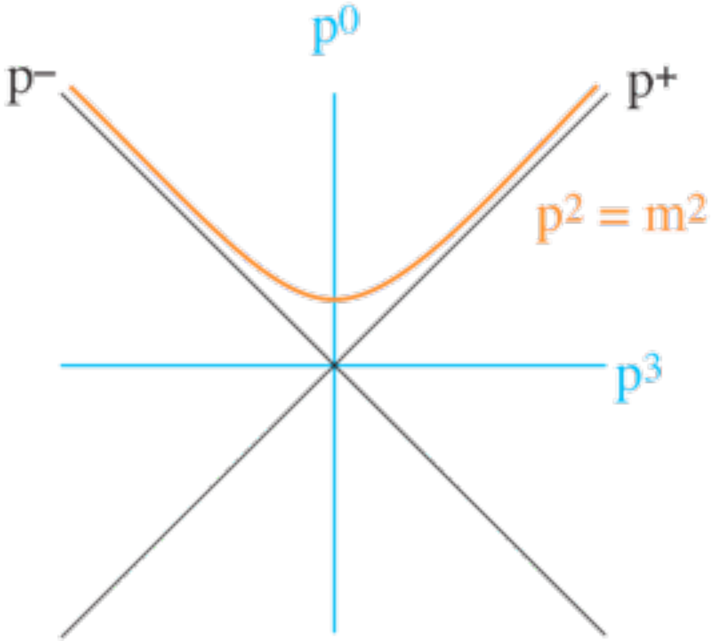

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Fourier transform

momentum space ⟷ coordinate space

$$e^{ip\cdot x}$$

$$p \cdot x = p^+ x^- + p^- x^+ - \vec{p}_T \cdot \vec{x}_T$$

--> $x^-$ is conjugate to $p^+$ and $x^+$ is conjugate to $p^-$

# space-time picture of the singularities

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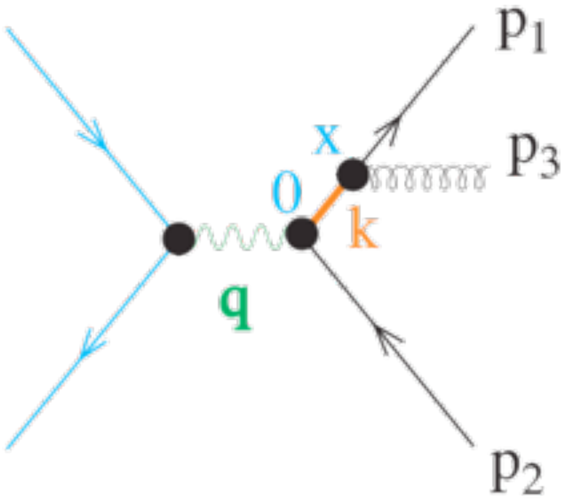

- define $k \equiv p_1 + p_3$
- use coordinates with $k^+$ large and $k_T = 0$
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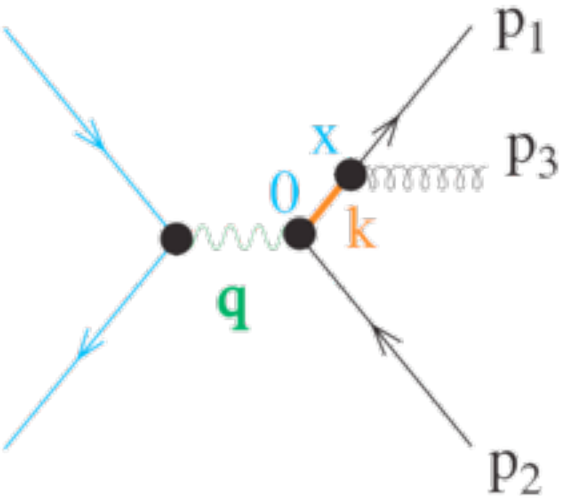

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**How far does the internal on-shell parton travel in space-time?**

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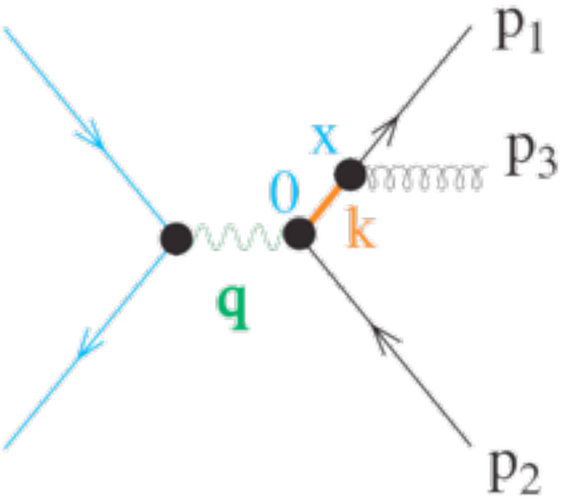

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- use coordinates with $k^+$ large and $k_T = 0$
- $k^2 = 2\, k^+ k^- \simeq 0$ corresponds to soft/collinear limit $\rightarrow$ $k^-$ small

**How far does the internal on-shell parton travel in space-time?**

$$k^+ \simeq \sqrt{s}/2$$

large

$$k^- \simeq (\vec{k}_T^2 + k^2)/\sqrt{s}$$

small

$\downarrow$ Fourier

$$x^+ \simeq 1/k^-$$

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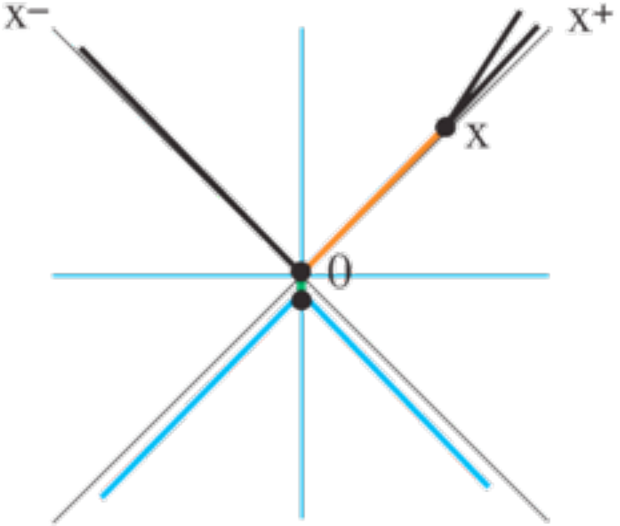

# space-time picture of the singularities

**What does this imply for our propagator going on-shell?**

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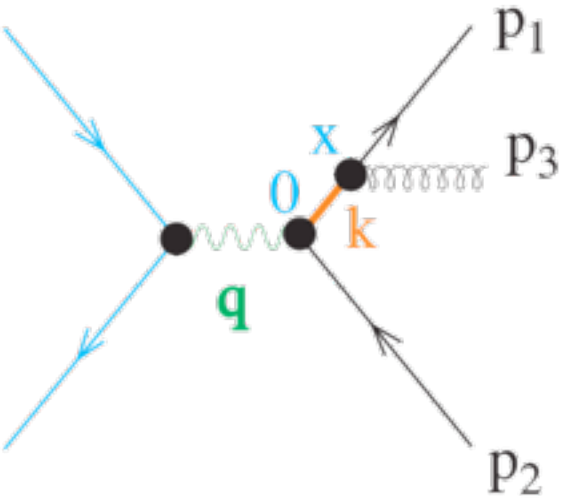

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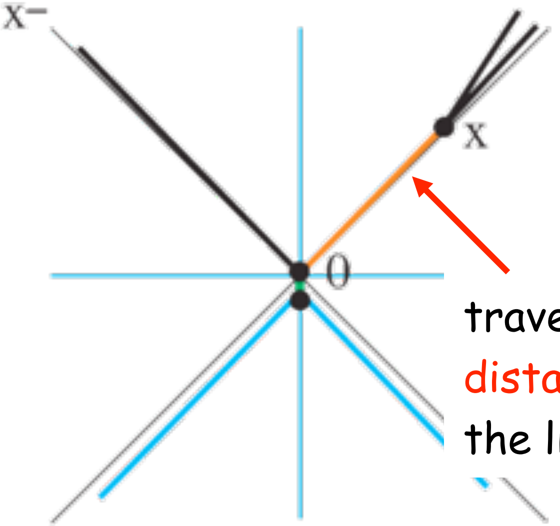

travels a long distance along the light-cone

**upshot:** soft/collinear singularities arise from interactions that happen a long time after the creation of the quark/antiquark pair

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to answer this, we have to formulate the

**concept of infrared safety**

# infrared-safe observables

**formal definition of infrared safety:**

Kunszt, Soper

study inclusive observables which do not distinguish between (n+1) partons and n partons in the soft/collinear limit, i.e., are insensitive to what happens at long-distance

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$$\begin{aligned}\mathcal{I} &= \frac{1}{2!}\int d\Omega_2 \frac{d\sigma[2]}{d\Omega_2} \mathcal{S}_2(p_1,p_2)\\ &+ \frac{1}{3!}\int d\omega_2 dE_3 d\Omega_3 \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} \mathcal{S}_3(p_1,p_2,p_3)\\ &+ \ldots\end{aligned}$$

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infrared safe **iff** [for $\lambda=0$ (soft) and $0 < \lambda < 1$ (collinear)]

$$\mathcal{S}_{n+1}(p_1,\ldots,(1-\lambda)p_n,\lambda p_n) = \mathcal{S}_n(p_1,\ldots,p_n)$$

# physics behind formal IR safety requirement

cannot resolve soft and collinear partons experimentally

→ intuitively reasonable that a theoretical calculation can be infrared safe as long as it is insensitive to long-distance physics (not a priori guaranteed though)

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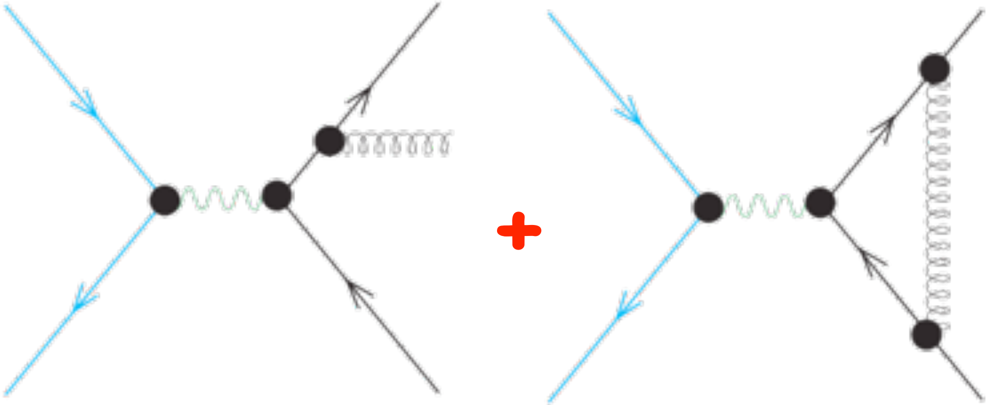
at a level of a pQCD calculation (e.g. $e^+e^-$ at $O(\alpha_s)$, i.e., n=2)

$$\mathcal{S}_{n+1}(p_1,\ldots,(1-\lambda)p_n,\lambda p_n) = \mathcal{S}_n(p_1,\ldots,p_n)$$

→ singularities of real gluon emission and virtual corrections cancel in the sum



extension of famous theorems by Kinoshita-Lee-Nauenberg and Bloch-Nordsieck

# example I: total cross section $e^+e^- \rightarrow$ hadrons

**simplest case:**

$$\boxed{S_n(p_1, \ldots, p_n) = 1}$$

fully inclusive quantity ⟷ we don't care what happens at long-distance

- the produced partons will all hadronize with probability one
- we do not observe a specific type of hadron
  (i.e. sum over a complete set of states)
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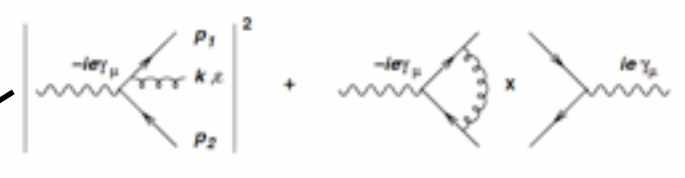
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum e_q^2\,(1+\Delta_{\text{QCD}})$$



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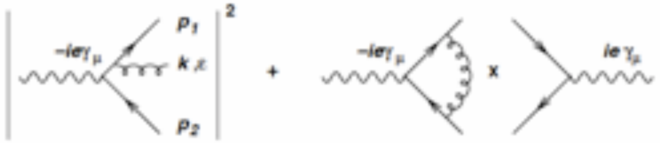
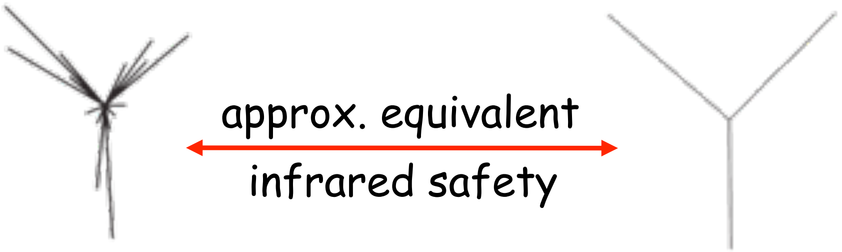


need to add up real and virtual corrections

**not IR safe:**

- energy of hardest gluon in event
- multiplicity of gluons or 1-gluon cross section

# example II: n-jet cross section



**experiment**

**QCD theory**

approx. equivalent

infrared safety

real physical event
with 3 hadron-jets

theor. jet event
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**jets are the central link between theory and experiment**

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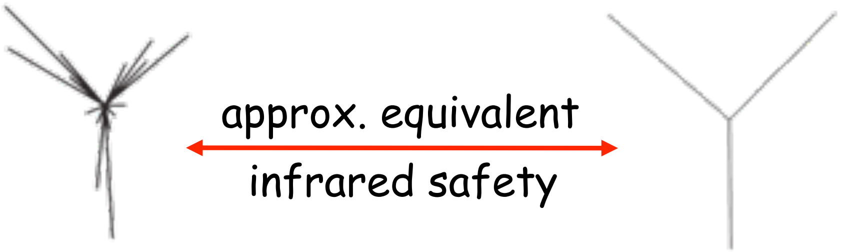

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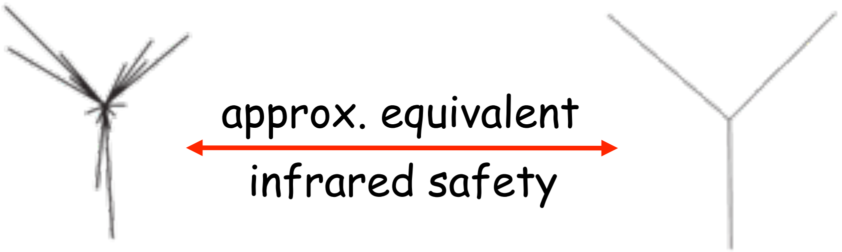

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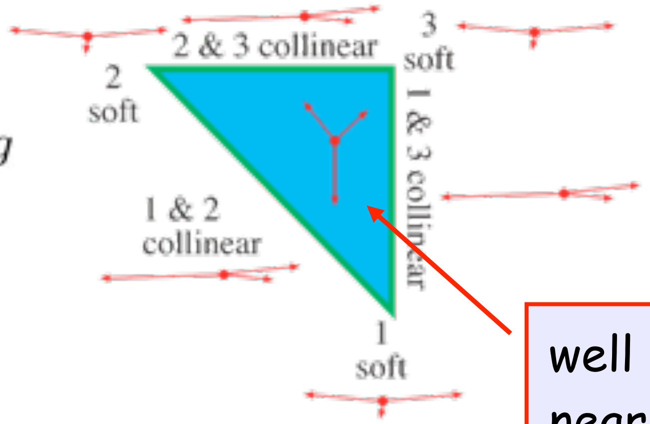
$e^+e^- \to q\bar{q}g$



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classify the final-state of hadrons (exp.) or partons (th.) according to the number of jets

well inside: 3-jets

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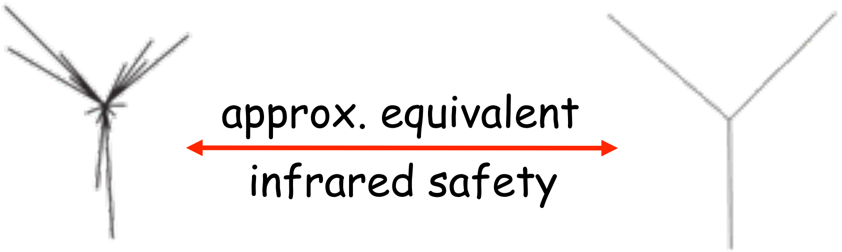

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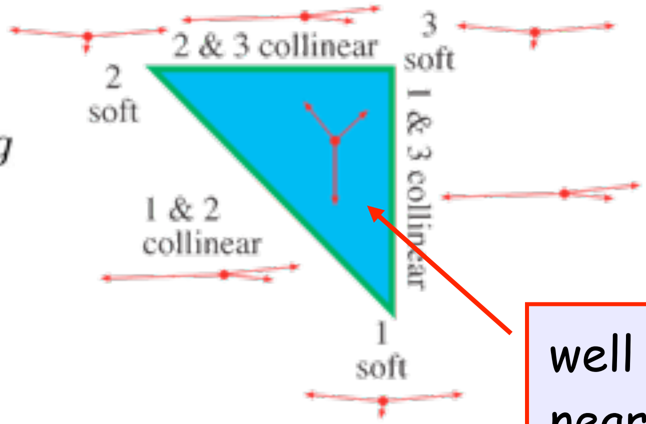

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**"2 or 3" depends on algorithm**

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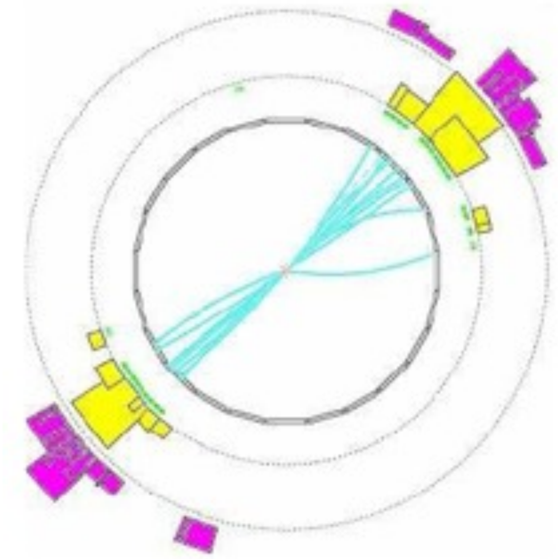

clearly (?) a 2-jet event

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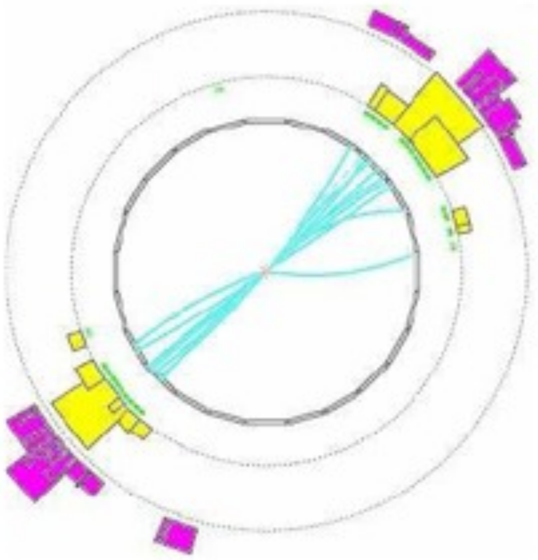

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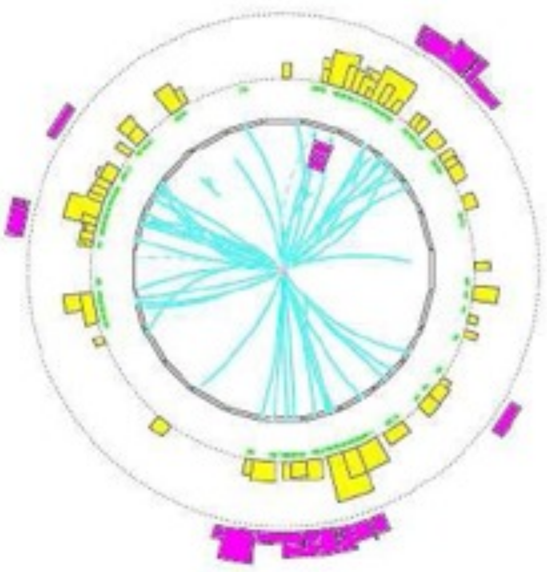

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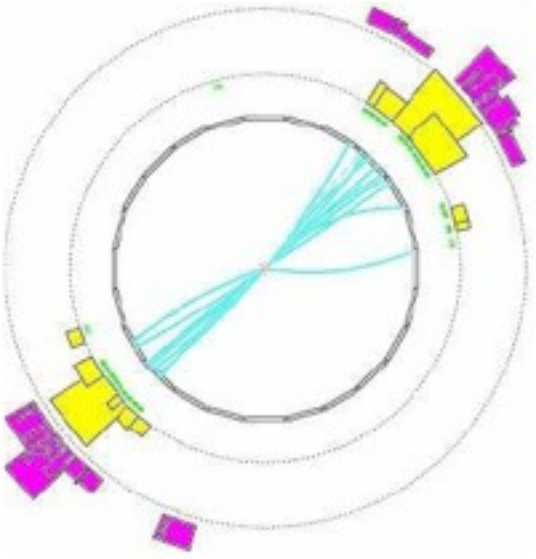

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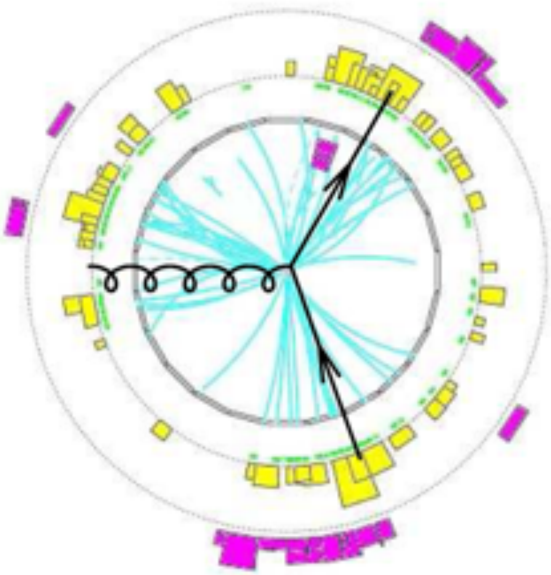

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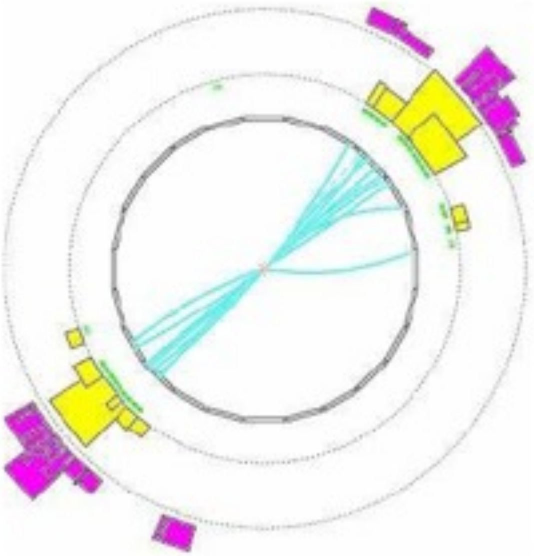

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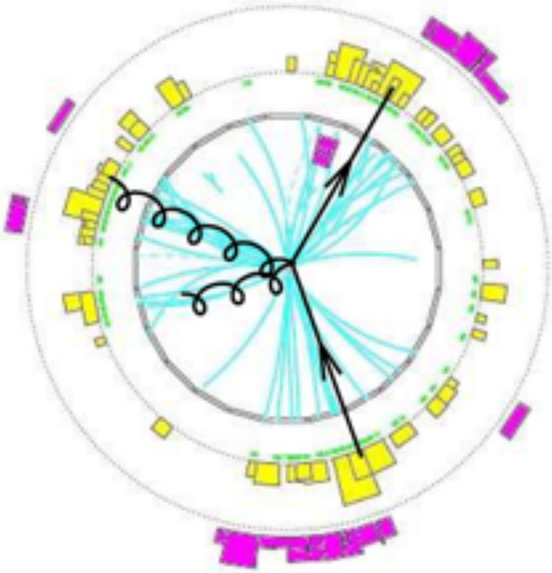

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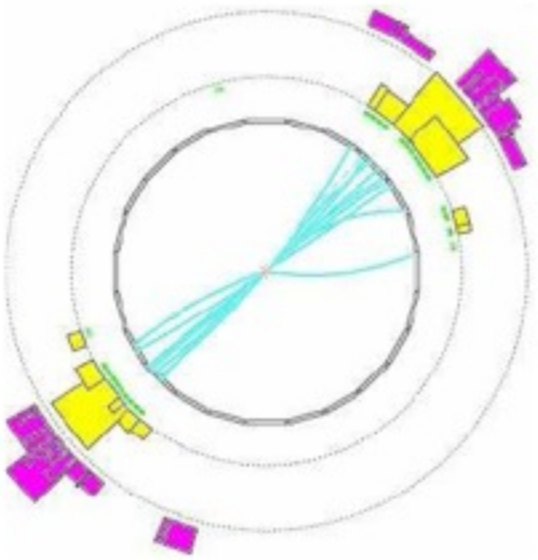

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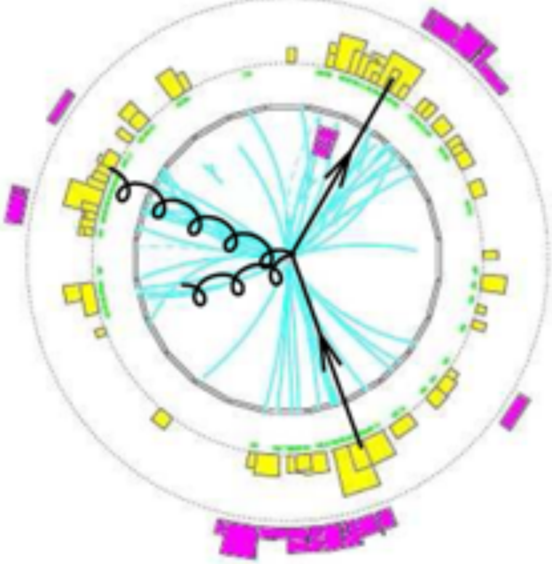

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the “best” jet definition does not exist – construction is unavoidably ambiguous

basically two issues:

- which particles/partons get put together in a jet → **jet algorithm**
- how to combine their momenta → **recombination scheme**

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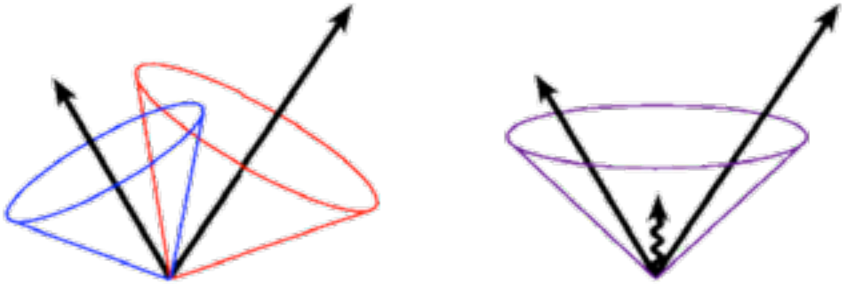

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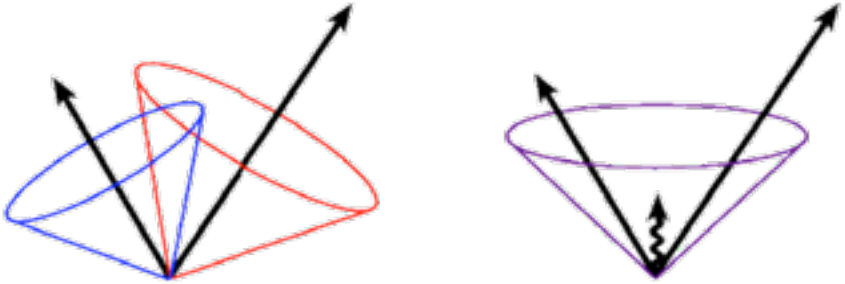

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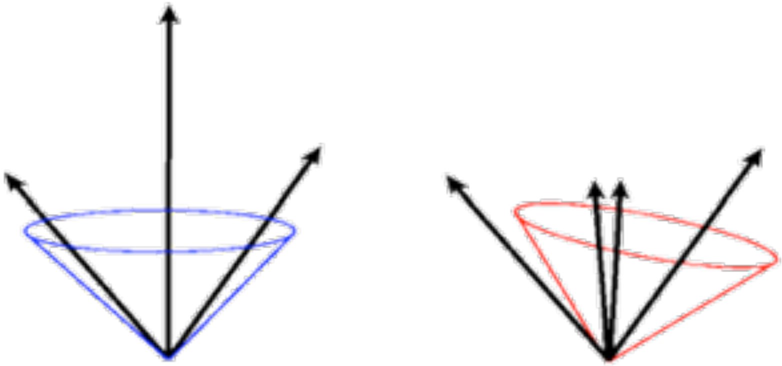

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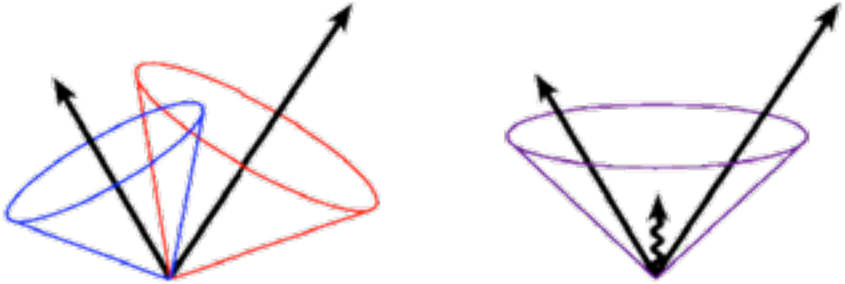

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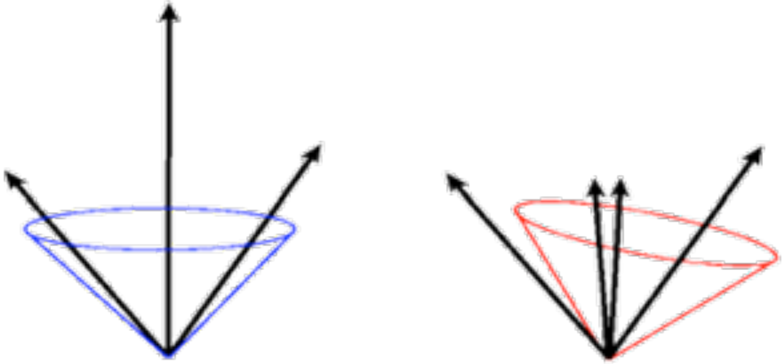

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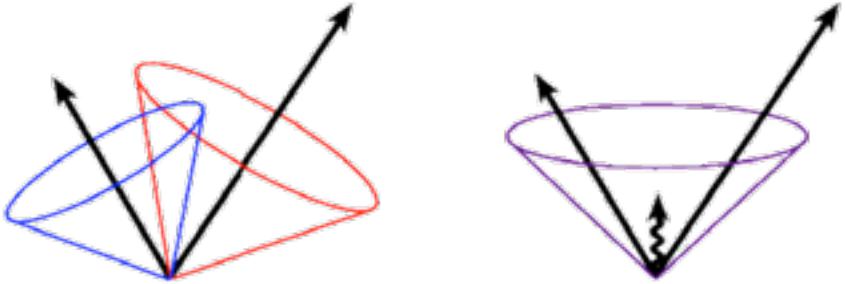

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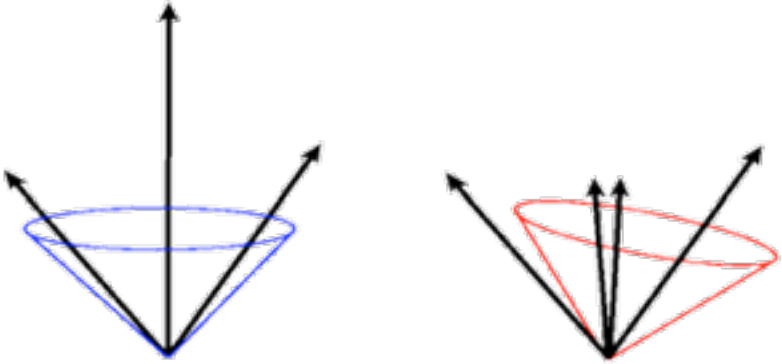
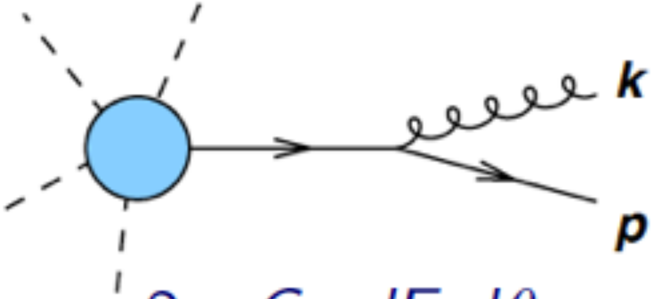


**(anti-) $k_T$ algorithms are the method of choice these days**

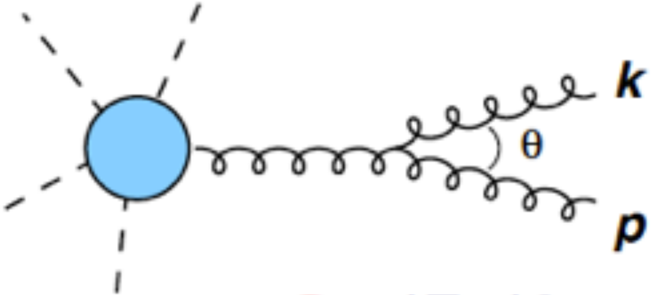
Cacciari, Salam, Soyez (FastJet tool)

# idea behind parton shower MC programs

- we have seen that emission of soft/collinear partons is favored
- we know exactly how and when it occurs (process-independent)



$$\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$



$$\frac{2\alpha_s C_A}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$

**this will provide the basis for a "parton shower"**

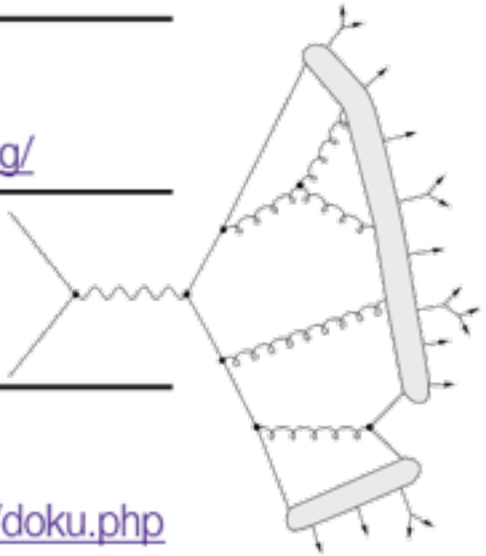
- **main idea:** seek for an approx. result such that soft/collinear

  enhanced terms are included to all orders

  emissions are probabilistic (needed to set up an event generator)

# popular parton shower programs

|  |  |  |
|---|---|---|
| PYTHIA | T. Sjöstrand et al. | http://home.thep.lu.se/~torbjorn/Pythia.html |
| HERWIG | G. Corcella et al. | http://hepwww.rl.ac.uk/theory/seymour/herwig/ |
| HERWIG++ | S. Gieseke et al. | http://projects.hepforge.org/herwig/ |
| SHERPA | F. Krauss et al. | http://projects.hepforge.org/sherpa/dokuwiki/doku.php |
| ISAJET | H. Baer et al. | http://www.nhn.ou.edu/~isajet/ |



- fail in high-multiplicity events or when large-angle emissions are relevant
- do better than fixed order calculations at lowish scales
- matching with NLO matrix elements well advanced: MC@NLO, POWHEG, …

# summary so far

pQCD cannot give all the answers
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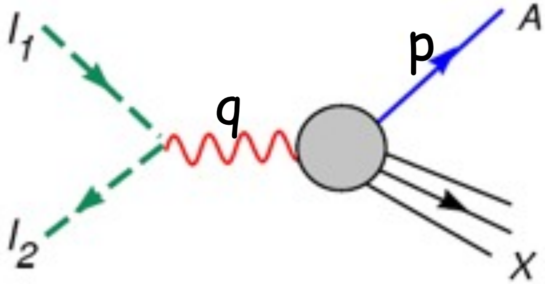
the **concept of factorization** will allow us to compute cross sections for a much wider class of processes than considered so far (involving hadrons in the initial and/or final state)

HERA, TeVatron, JLab, RHIC, LHC, ..., EIC

# hadrons: a new “long distance problem”

consider the one-particle inclusive cross section:



identified hadron

e.g. (A = π)

not measured

$$\boxed{\frac{d\sigma(e^{+}e^{-}\rightarrow \pi + X)}{dE_{\pi}}}$$

**not infrared safe by itself!**

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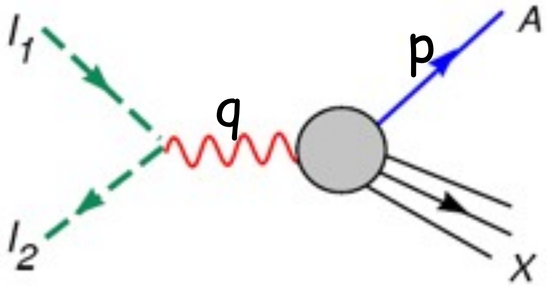

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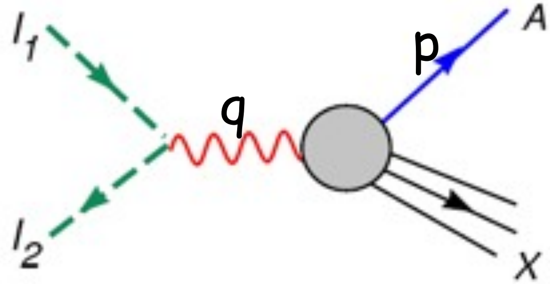

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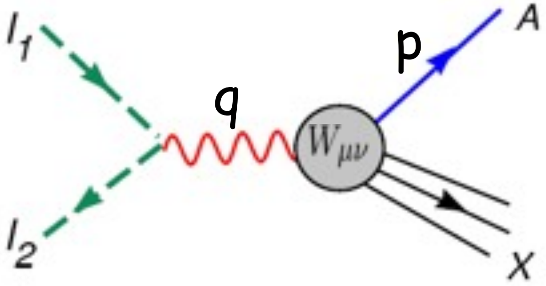
**general feature of QCD processes with observed (=identified) hadrons in the initial and/or final state**

# factorization

**strategy**: try to factorize the physical observable into a calculable infrared safe and a non-calculable but universal piece

**how does it work?**



leptonic tensor

$$d\sigma = \frac{4\alpha^2}{sQ^2}\frac{d^3\vec{p}}{2|\vec{p}|}L^{\mu\nu}W_{\mu\nu}$$

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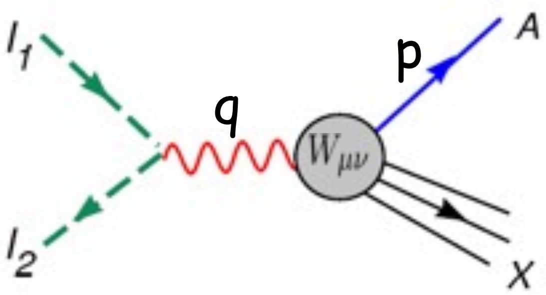

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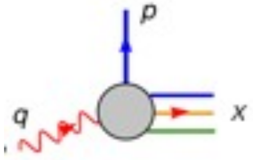
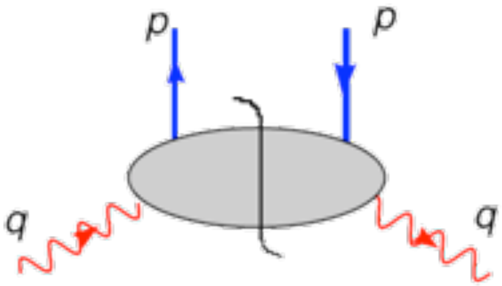
**hadronic tensor**

**hadronic tensor $W_{\mu\nu}$:**



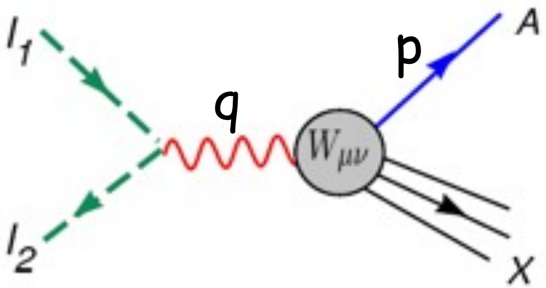
square of the hadronic scattering amplitude summed over all final-states X except A(p)



# factorization

**strategy**: try to factorize the physical observable into a calculable infrared safe and a non-calculable but universal piece

**how does it work?**

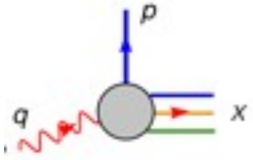


$$d\sigma = \frac{4\alpha^2}{sQ^2}\frac{d^3\vec{p}}{2|\vec{p}|}L^{\mu\nu}W_{\mu\nu}$$

**leptonic tensor**

**hadronic tensor**

**hadronic tensor $W_{\mu\nu}$:**



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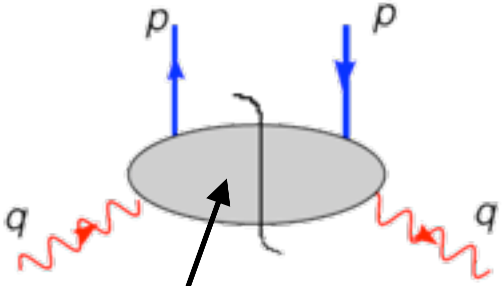

**need to factorize long-distance physics**

# concept of factorization - pictorial sketch

**factorization** = isolating and absorbing infrared singularities accompanying observed hadrons

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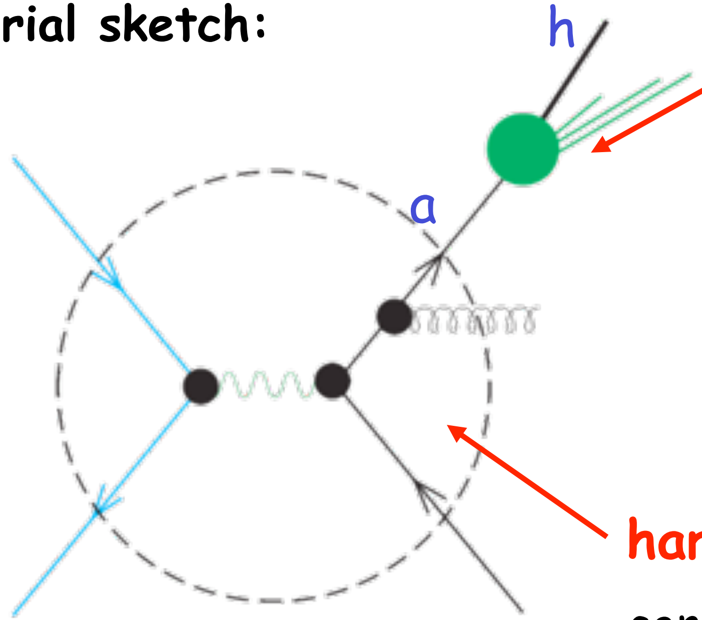
**pictorial sketch:**



**fragmentation functions** $D_a^h$

contains all **long-distance** interactions
hence not calculable but universal

**physical interpretation:**

probability to find a hadron carrying
a certain momentum of parent parton

**hard scattering** $\widehat{F}_a$

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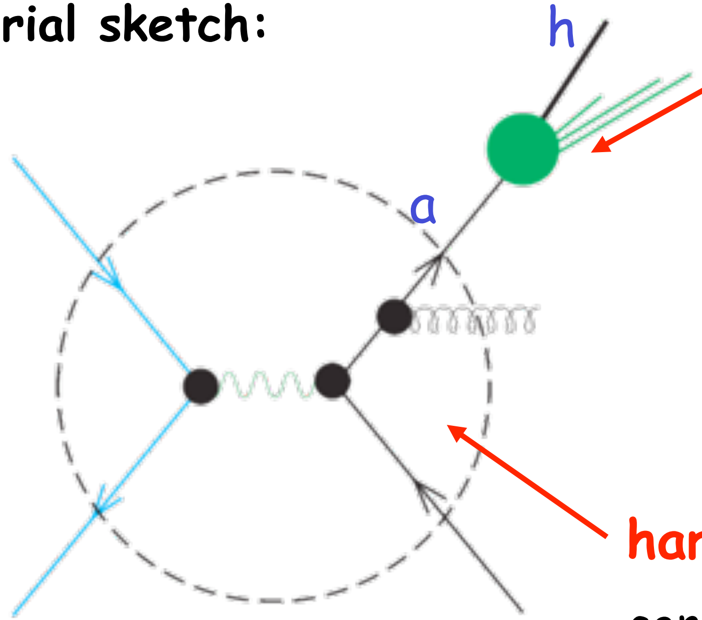

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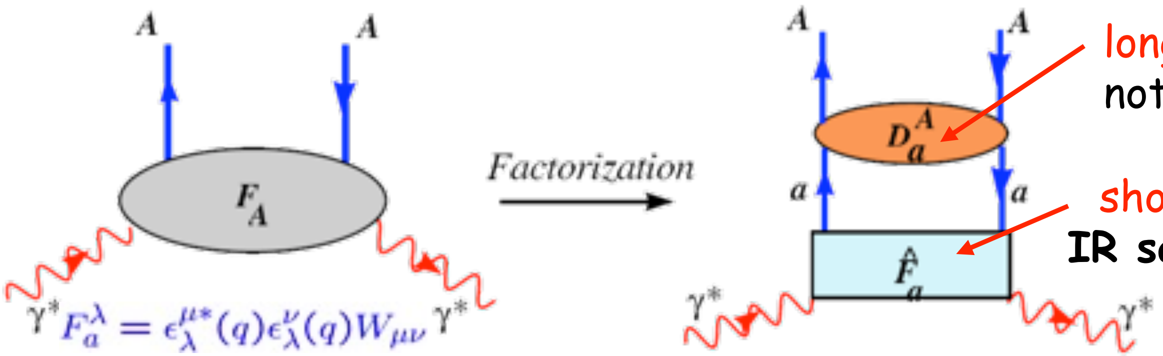
**aside:** fragmentation fcts. play an important role in learning about nucleon (spin) structure from semi-inclusive DIS data by COMPASS & HERMES or from hadron production at RHIC

# factorization - detailed picture

**more explicitly**



$F_a^\lambda = \epsilon_\lambda^{\mu *}(q)\epsilon_\lambda^\nu(q) W_{\mu\nu}$

long-distance
not calculable

short distance
**IR safe**, calculable

$\lambda$=L,T (pol. of $\gamma^*$)

$$\frac{d\sigma}{dz d\cos\theta} = \frac{\pi\alpha^2}{2s}\left[F_A^T(z,Q)(1+\cos^2\theta) + F_A^L(z,Q)\sin^2\theta\right]$$

where

$$F_A^{T,L}(z,Q) = \sum_a \hat{F}_a^{T,L}(z,\frac{Q}{\mu_f}) \otimes D_a^h(z,\mu_f)$$

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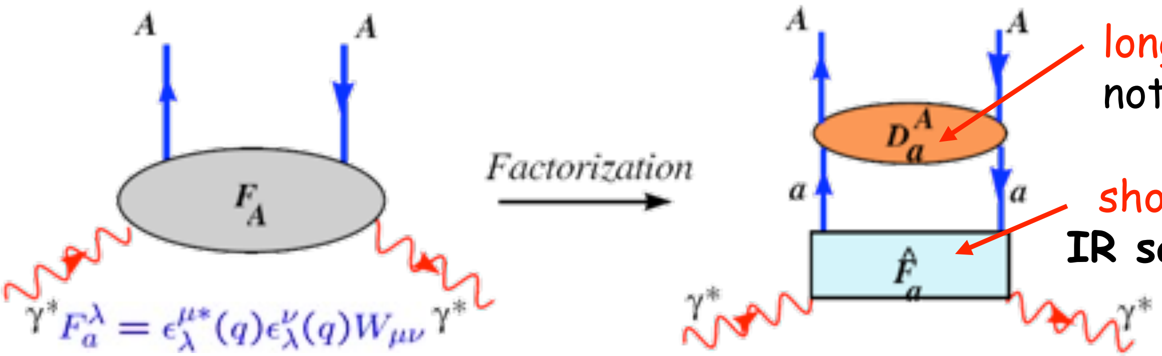

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$$\gamma^* F_a^\lambda = \epsilon_\lambda^{\mu *}(q)\epsilon_\lambda^\nu(q)W_{\mu\nu}\, \gamma^*$$

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factorization scale (arbitrary!)

characterizes the boundary between
short and long-distance physics

**physics indep. of $\mu_f$ → renormalization group**

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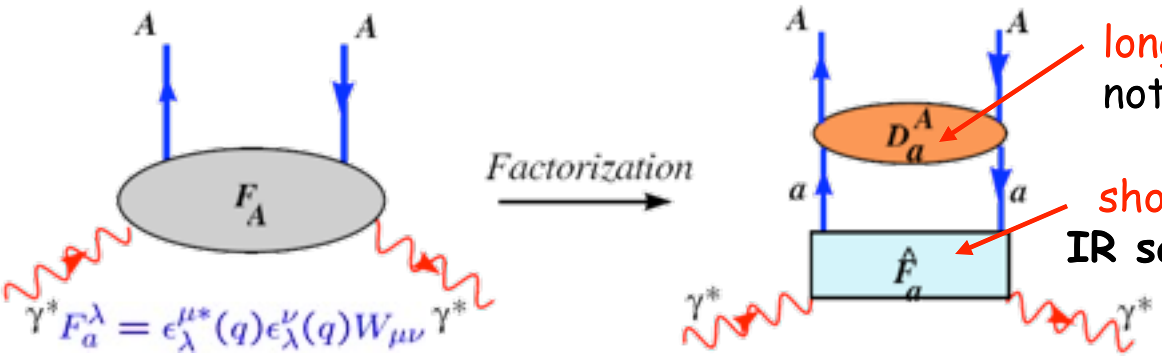

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"convolution"

$$f(x) \otimes g(x) \equiv \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y)$$

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# factorization - detailed picture

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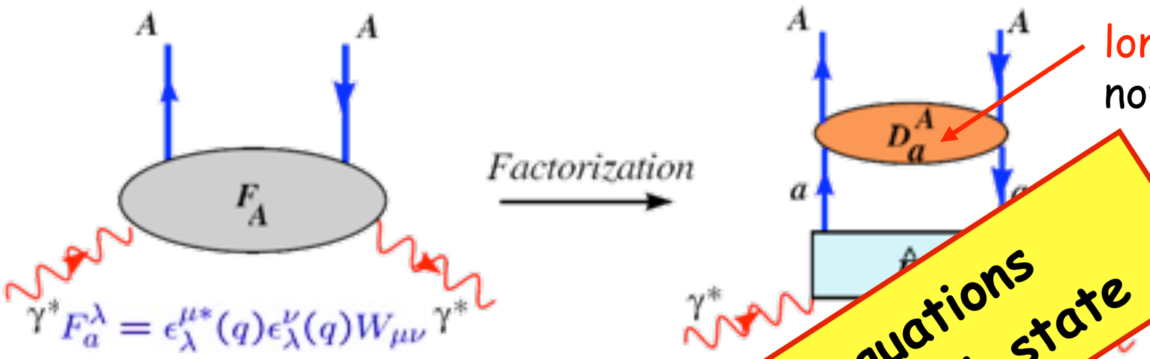

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**before studying renormalization group equations**
**let's first introduce hadrons also in the initial-state**

"conv

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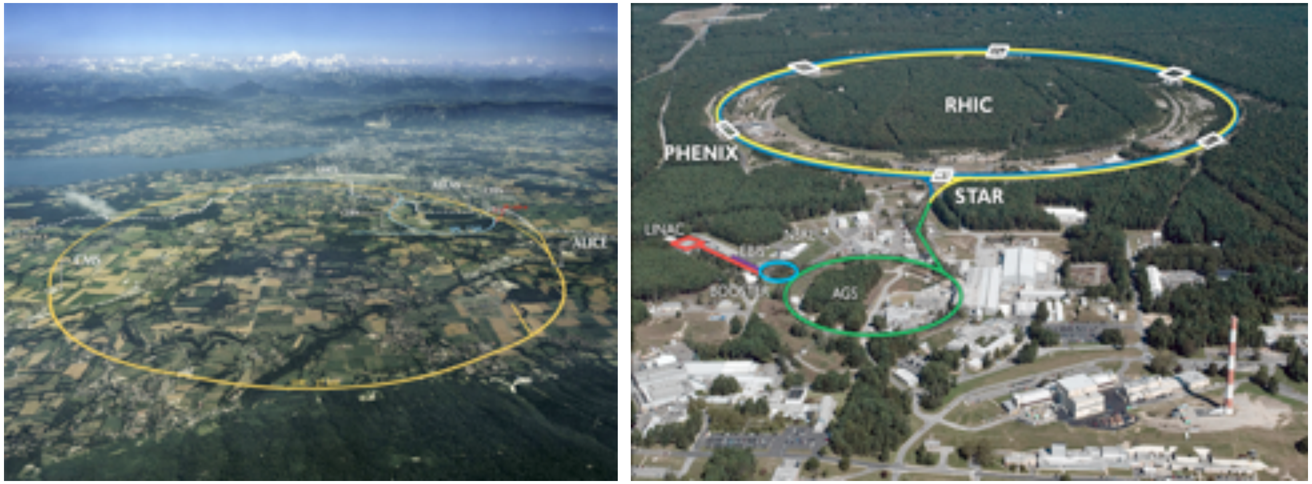
# take home message for part II

## the QCD toolbox

- QCD is a non-Abelian gauge theory: gluons are self-interacting $\rightarrow$ asymptotic freedom (large Q), confinement (small Q)
- QCD calculations are singular when any two partons become collinear or a gluon becomes soft; basis for parton shower MCs
- choose infrared/collinear safe observables for comparison between experiment and perturbative QCD
- jets (= cluster of partons): best link between theory and exp.; needs a proper IR safe jet definition in theory and experiment
- factorization allows to deal with hadronic processes introduces arbitrary scale -> leads to RGEs

early microscopes



the World's most powerful microscopes

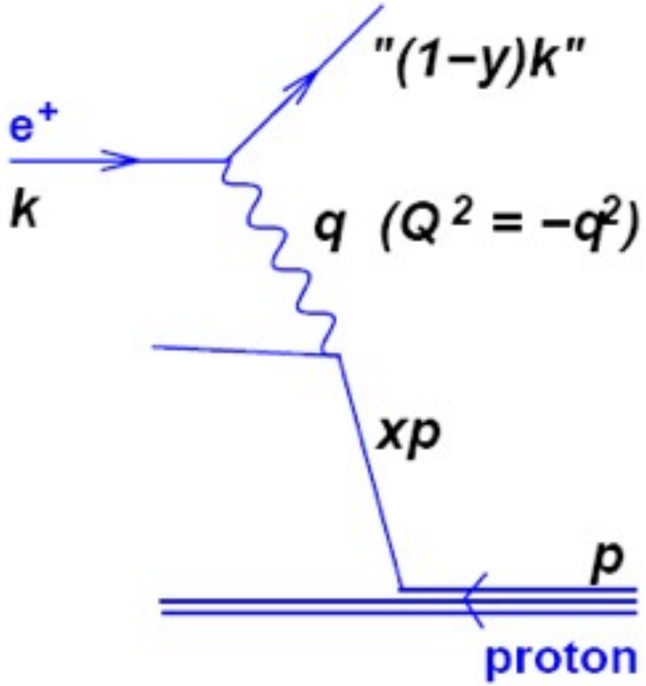
# Part III

## inward bound: "femto-spectroscopy"

QCD initial state, partons, DIS, factorization, renormalization group, hadron-hadron collisions

# partons in the initial state: the DIS process

start with the simplest process: **deep-inelastic scattering**



**relevant kinematics:**

$$x = \frac{Q^2}{2p \cdot q} \qquad y = \frac{p \cdot q}{p \cdot k} \qquad Q^2 = xys$$

- $Q^2$: photon virtuality ↔ resolution r∼1/Q at which the proton is probed
- x: long. momentum fraction of struck parton in the proton
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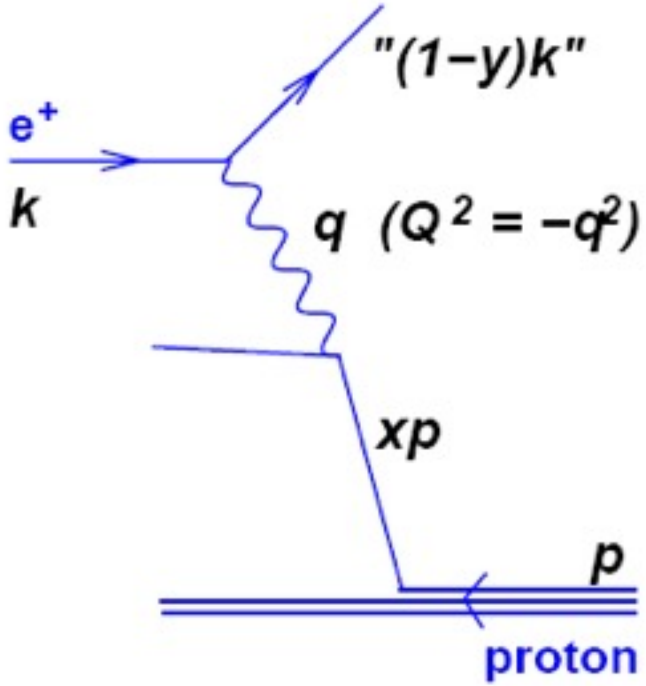

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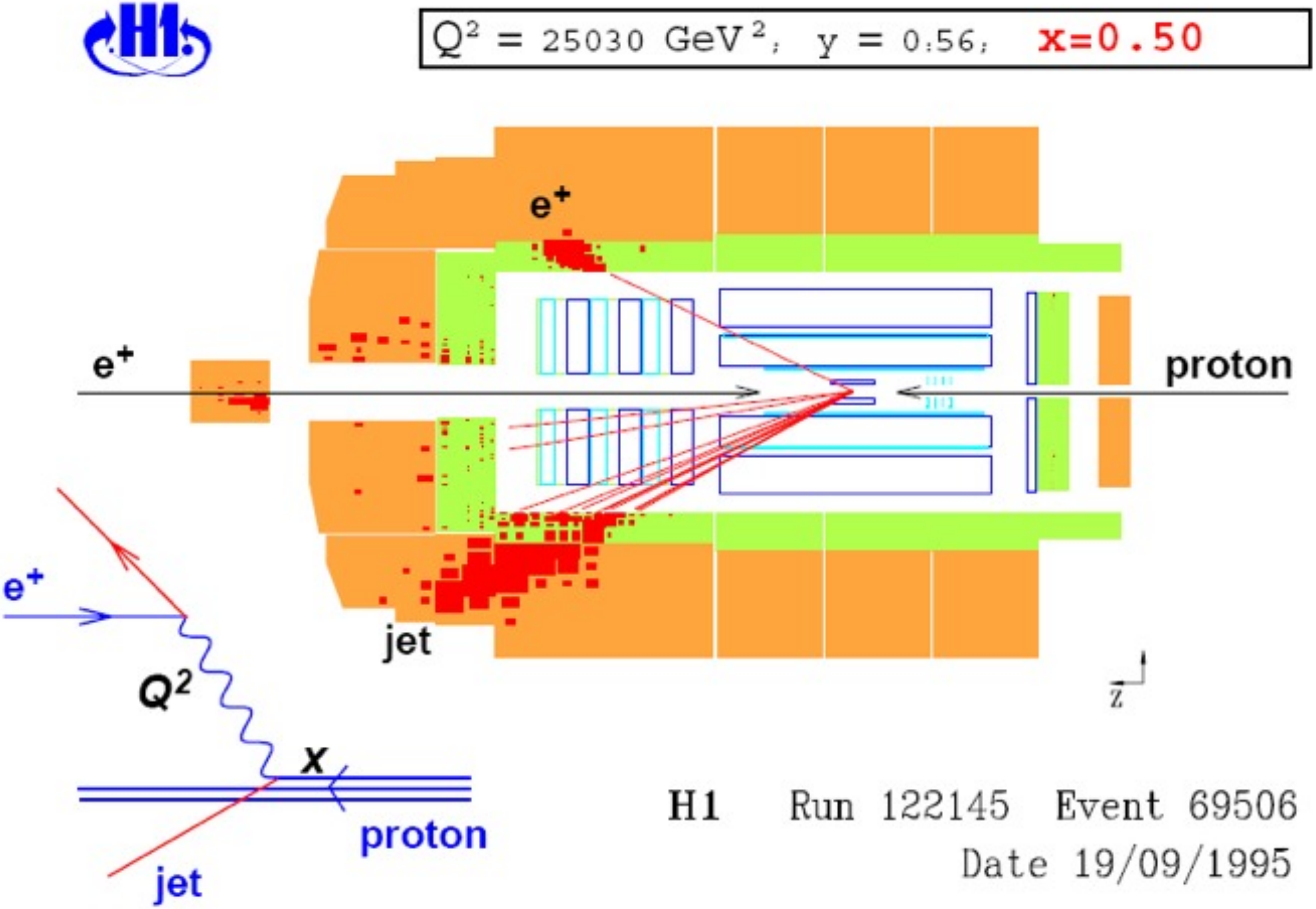
"deep-inelastic": $Q^2 \gg 1\ \mathrm{GeV}^2$

"scaling limit": $Q^2 \to \infty$, x fixed

resolution: $\frac{\hbar}{Q} \approx \frac{2 \times 10^{-16}\mathrm{m}}{Q[\mathrm{GeV}]}$

$r \sim 1/Q$

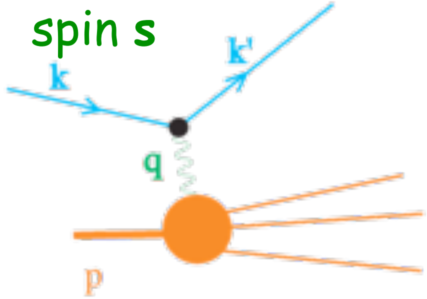
# a typical DIS event



# analysis of DIS: 1st steps

electroweak theory tells us how the virtual vector boson (here $\gamma^*$) couples:

spin s



spin S

$$d\sigma = \frac{4\alpha^2}{s}\frac{d^3\vec{k}'}{2|\vec{k}'|}\frac{1}{Q^4}L^{\mu\nu}(k,q,s)\boxed{W_{\mu\nu}(p,q,S)}$$

leptonic tensor from QED

**hadronic tensor** contains information about hadronic structure

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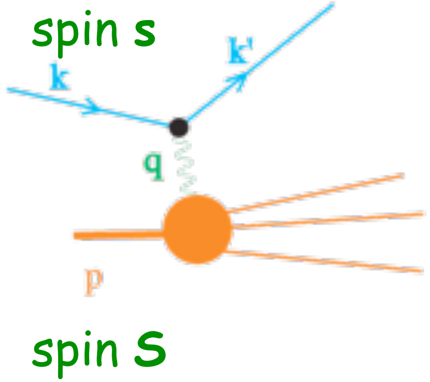

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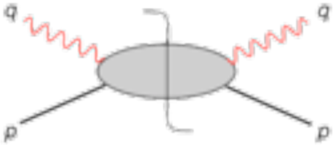
parity & Lorentz inv., hermiticity $W^{\nu\mu}=W^{\mu\nu *}$, current conservation $q_\mu W^{\mu\nu}=0$ **dictate**:

$$W^{\mu\nu}(P,q,S) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\,\langle P,S|\,J_\mu(z)\,J_\nu(0)\,|P,S\rangle$$



$$= \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)F_1(x,Q^2) + \left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)F_2(x,Q^2)$$

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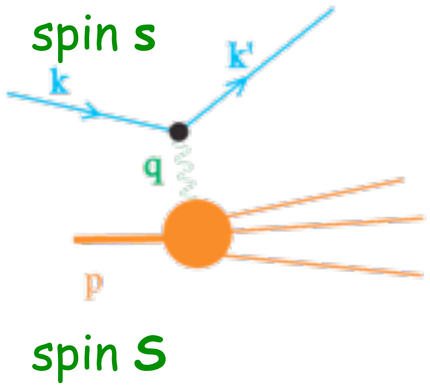

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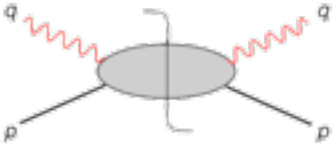

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**unpol.structure fcts.$F_{1,2}$**

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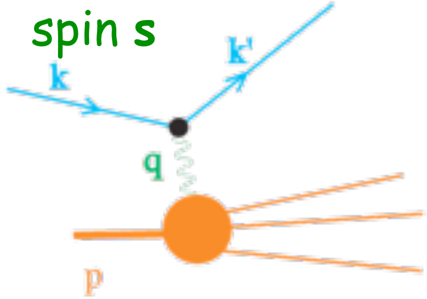

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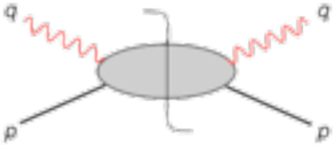

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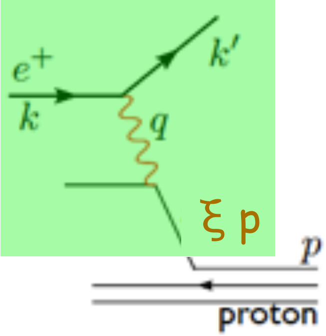
**unpol. structure fcts. $F_{1,2}$**

**pol. structure fcts. $g_{1,2}$ – measure W(P,q,S) – W(P,q,-S) !**

# DIS in the naïve parton model

**let's do a quick calculation**: consider **electron-quark scattering**



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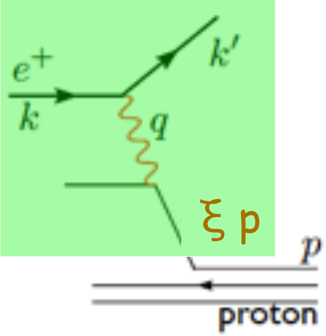

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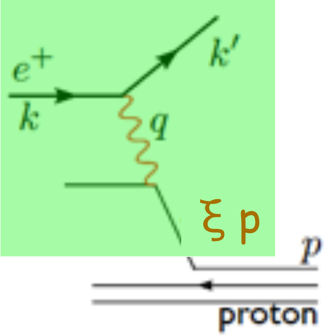

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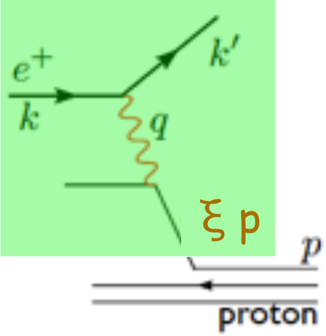

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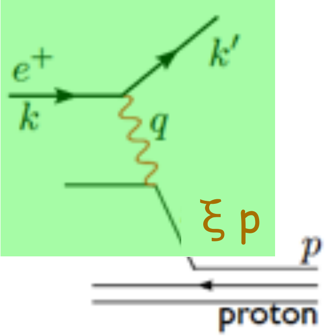

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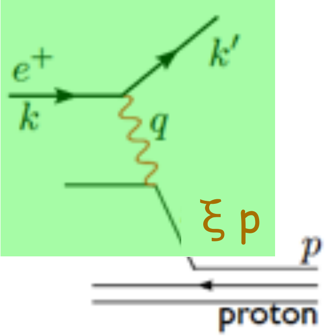

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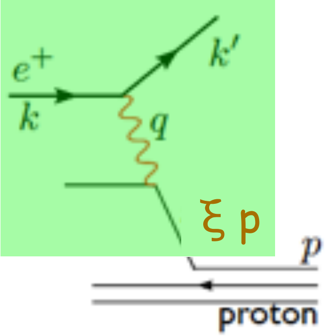

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to obtain

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}[1 + (1 - y)^2]$$

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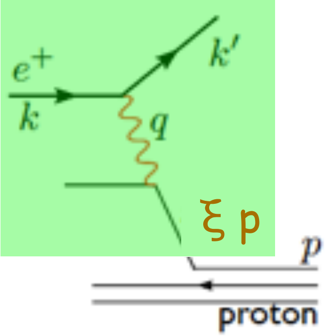

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$$\hat{u} = (p_q-k')^2$$

next: express by usual DIS variables

$x = \frac{Q^2}{2p\cdot q}$ $\quad y = \frac{p\cdot q}{p\cdot k}$ $\quad Q^2 = xys$ find

$$\hat{s} = \xi Q^2/(xy) = \xi\, s$$
$$\hat{t} = q^2 = -Q^2$$
$$\hat{u} = \hat{s}\,(y-1)$$

and use the massless 2->2 cross section

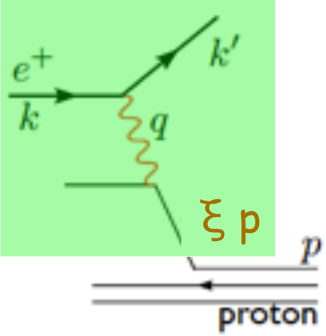
$\frac{d\sigma}{dt} = \frac{1}{16\pi\hat{s}^2}\overline{\sum}|\mathcal{M}|^2$ to obtain $\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}[1+(1-y)^2]$

next: use on-mass shell constraint

$$p_q'^2 = (p_q+q)^2 = q^2 + 2p_q\cdot q \qquad = -2p\cdot q\,(x-\xi) = 0$$

this implies that ξ is equal to Bjorken x

# DIS in the naïve parton model



**let's do a quick calculation**: consider **electron-quark scattering**

find $\overline{\sum}|\mathcal{M}|^2 = 2e_q^2e^4\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$ with the usual Mandelstam's

$$\hat{s}=(k+p_q)^2$$
$$\hat{t}=(k-k')^2$$
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$x=\frac{Q^2}{2p\cdot q}$ $y=\frac{p\cdot q}{p\cdot k}$ $Q^2=xys$ find

$$\hat{s}=\xi Q^2/(xy)=\xi\, s$$
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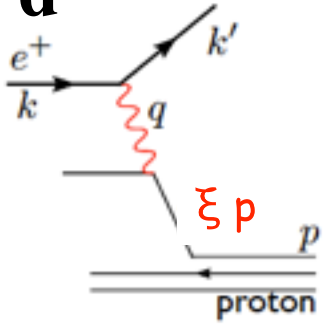
this implies that ξ is equal to Bjorken x

to obtain $\frac{d\sigma}{dxdQ^2}=\frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi)$

# DIS in the naïve parton model cont'd



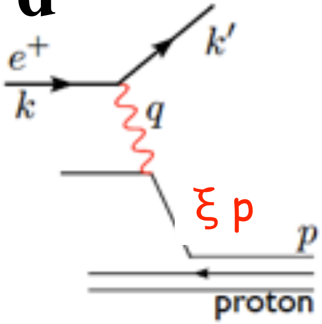
compare our result

$$\frac{d\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi)$$

to what one obtains with the hadronic tensor (on the quark level)

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[[1+(1-y)^2]F_1(x) + \frac{(1-y)}{x}(F_2(x) - 2xF_1(x))\right]$$

# DIS in the naïve parton model cont'd



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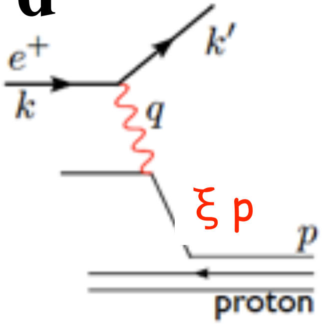
and read off

$$\boxed{F_2 = 2xF_1 = xe_q^2\,\delta(x-\xi)}$$

Callan Gross relation

reflects spin 1/2 nature of quarks

# DIS in the naïve parton model cont’d



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and read off

$$\boxed{\mathbf{F_2 = 2xF_1 = xe_q^2}\,\delta(\mathbf{x}-\xi)}$$

**Callan Gross relation**
reflects spin 1/2 nature of quarks

proton structure functions then obtained by weighting the quark str. fct. with the **parton distribution functions** (probability to find a quark with momentum ξ)

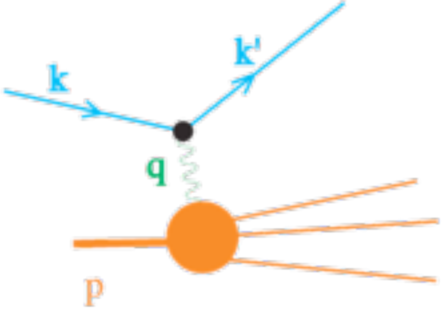
$$\mathbf{F_2 = 2xF_1} = \sum_{\mathbf{q,q'}}\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{d}\xi\,\mathbf{q}(\xi)\,\mathbf{xe_q^2}\,\delta(\mathbf{x}-\xi)$$

$$= \sum_{\mathbf{q,q'}} \mathbf{e_q^2\,x\,q(x)}$$

**DIS measures the charged-weighted sum of quarks and antiquarks**

**“scaling” - no dependence on scale Q**

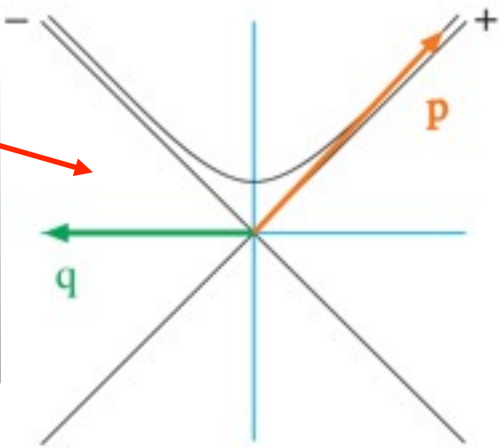
# space-time picture of DIS



this can be best understood in a reference frame where the proton moves very fast and Q>>$m_h$ is big

(recall light-cone kinematics from part II)

| 4-vector | hadron rest frame | Breit frame |
|---|---|---|
| $(p^+, p^-, \vec{p}_T)$ | $\frac{1}{\sqrt{2}}(m_h, m_h, \vec{0})$ | $\frac{1}{\sqrt{2}}(\frac{Q}{x}, \frac{xm_h^2}{Q}, \vec{0})$ |
| $(q^+, q^-, \vec{q}_T)$ | $\frac{1}{\sqrt{2}}(-m_hx, \frac{Q^2}{m_hx}, \vec{0})$ | $\frac{1}{\sqrt{2}}(-Q, Q, \vec{0})$ |



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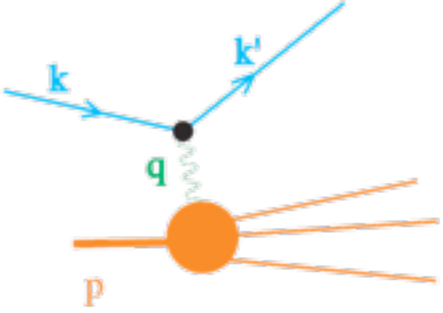

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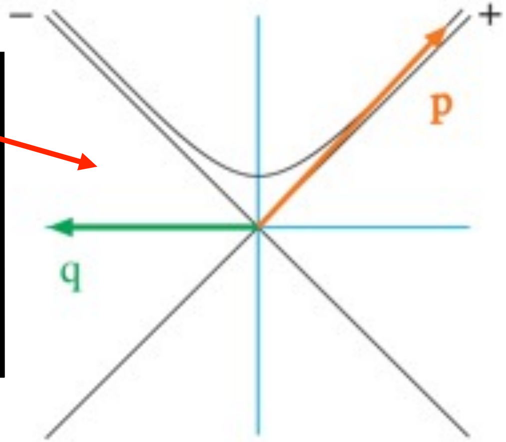
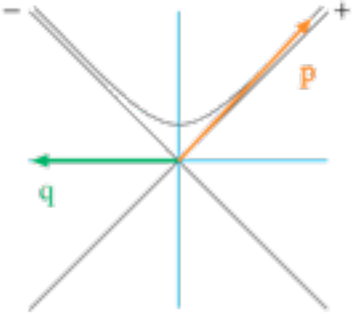


Lorentz boost

in general $(a^+, a^-, \vec{a}_T) \rightarrow (e^{\omega} a^+, e^{-\omega} a^-, \vec{a}_T) = (a'^+, a'^-, \vec{a}')$

here: $e^{\omega} = Q/(x m_h)$

# space-time picture of DIS – cont'd

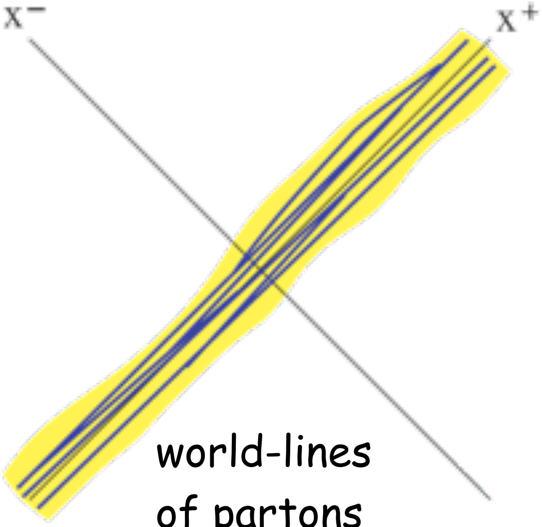


simple estimate for typical time-scale of interactions among the partons inside a fast-moving hadron:

**rest frame:** $\Delta x^+ \sim \Delta x^- \sim \frac{1}{m}$

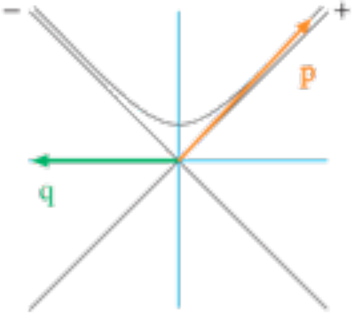
**Breit frame:** $\Delta x^+ \sim \frac{1}{m}\frac{Q}{m} = \frac{Q}{m^2}$ large

$\Delta x^- \sim \frac{1}{m}\frac{m}{Q} = \frac{1}{Q}$ small



world-lines of partons

# space-time picture of DIS – cont'd



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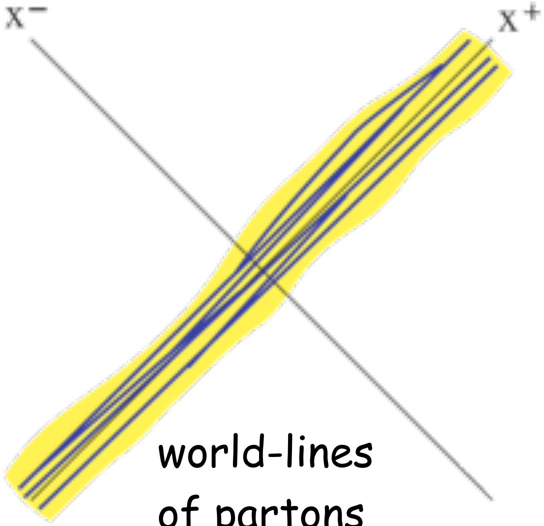
**interactions between partons are spread out inside a fast moving hadron**



world-lines of partons

# space-time picture of DIS – cont'd

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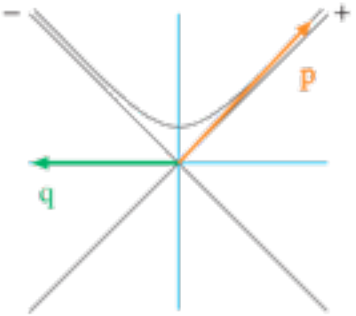

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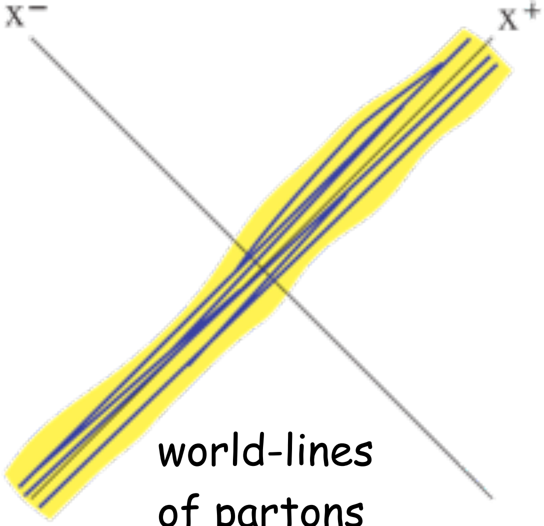


world-lines of partons

**interactions between partons are spread out inside a fast moving hadron**

How does this compare with the time-scale of the hard scattering?

# foundation of naïve Parton Model

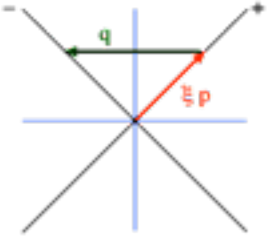
Feynman;
Bjorken, Paschos

**Breit frame:**

proton moves very fast and $Q \gg m_h$ is big

$$(p^+,p^-,\vec{p}_T)=\frac{1}{\sqrt{2}}(\frac{Q}{x},\frac{xm_h^2}{Q},\vec{0})\quad(q^+,q^-,\vec{q}_T)=\frac{1}{\sqrt{2}}(-Q,Q,\vec{0})$$

struck quark
on-shell



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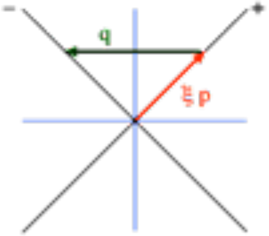

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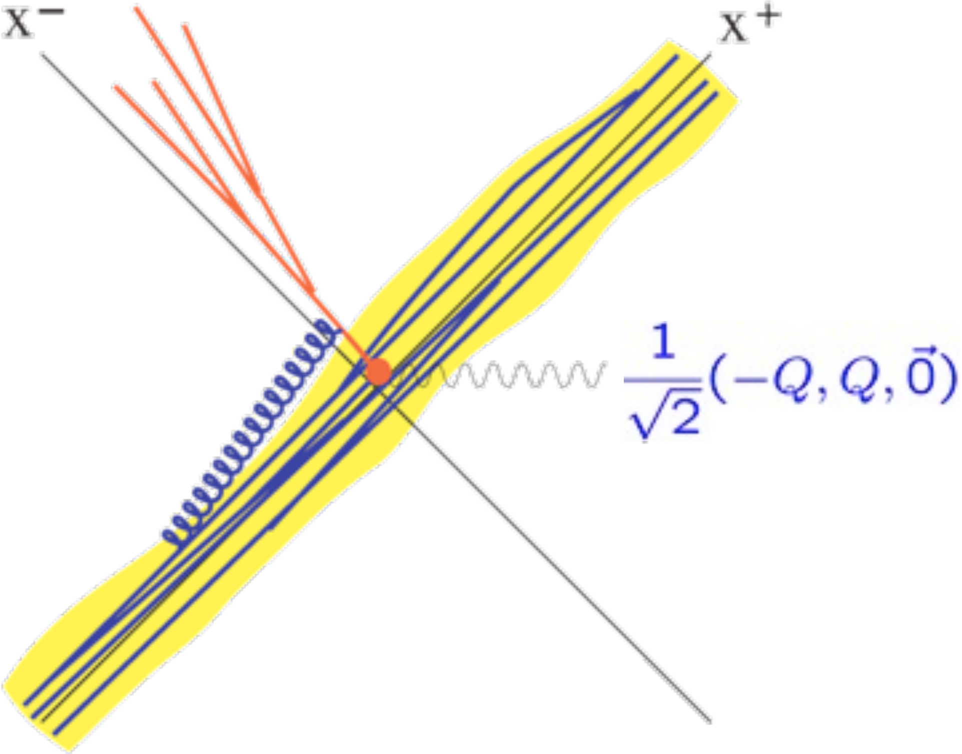

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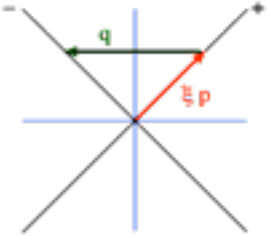

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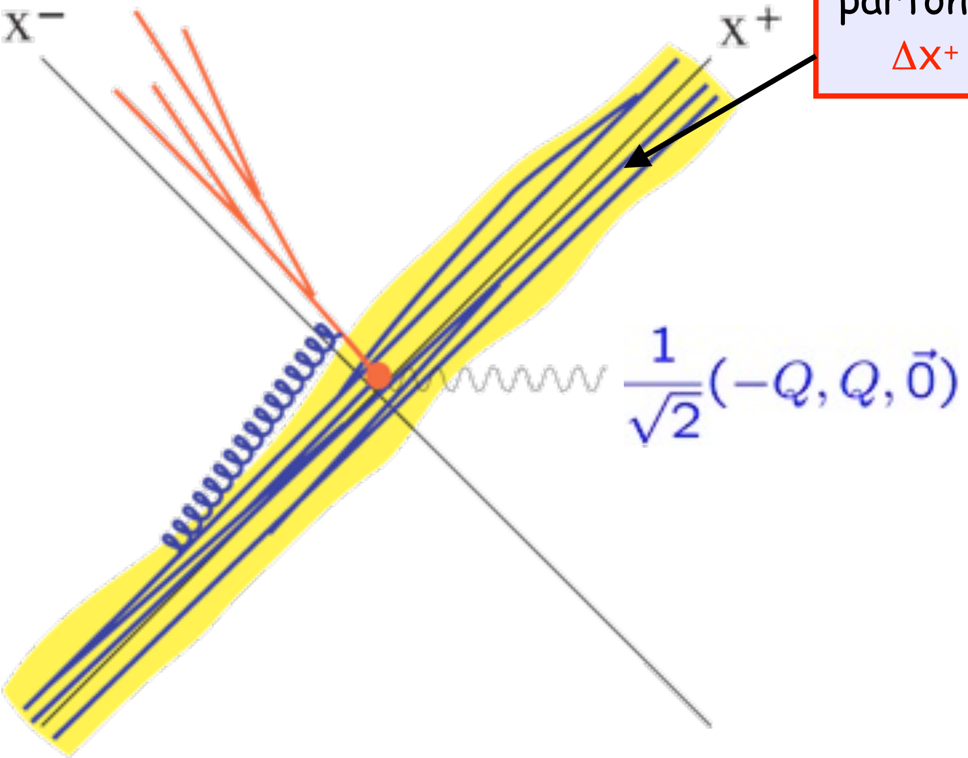

interactions of partons dilated
$\Delta x^+ \approx Q/m^2$

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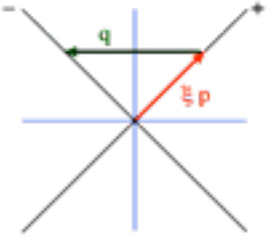

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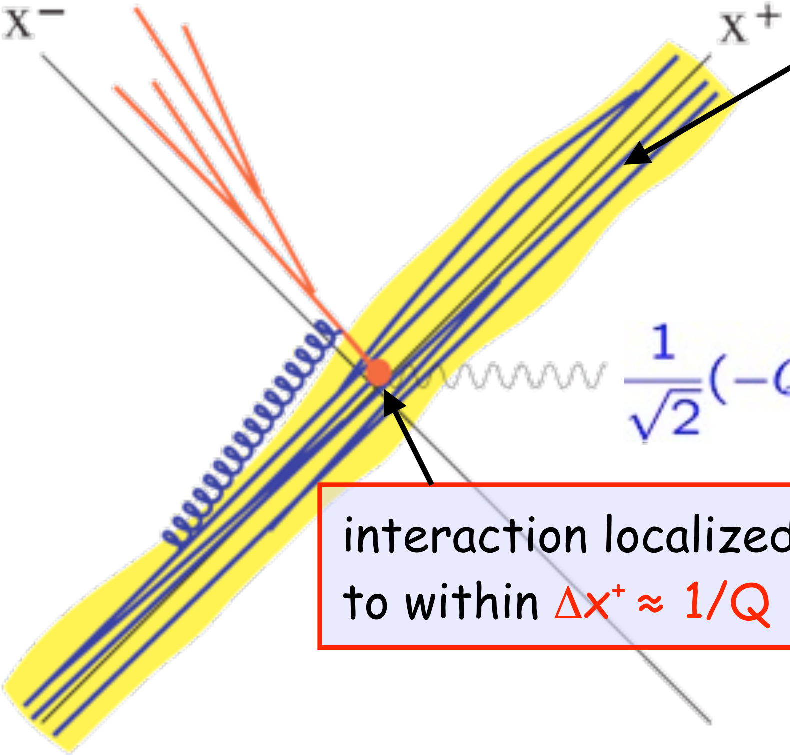

$$\frac{1}{\sqrt{2}}(-Q, Q, \vec{0})$$

interactions of partons dilated $\Delta x^+ \approx Q/m^2$

interaction localized to within $\Delta x^+ \approx 1/Q$

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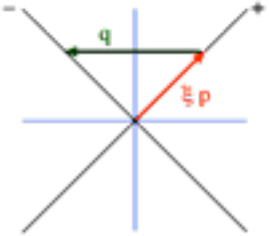

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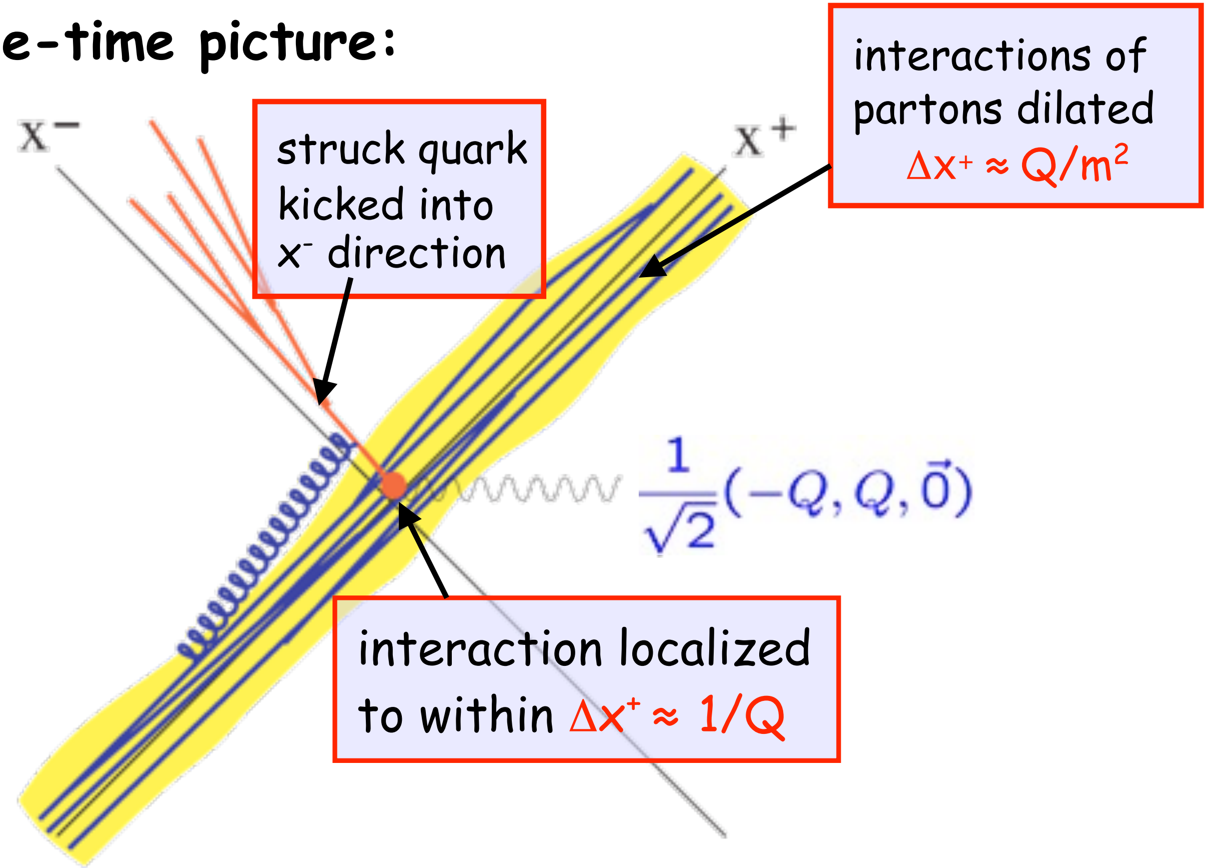

struck quark kicked into $x^-$ direction

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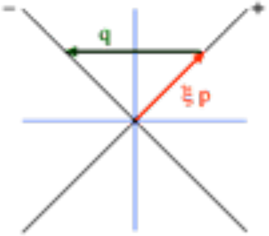

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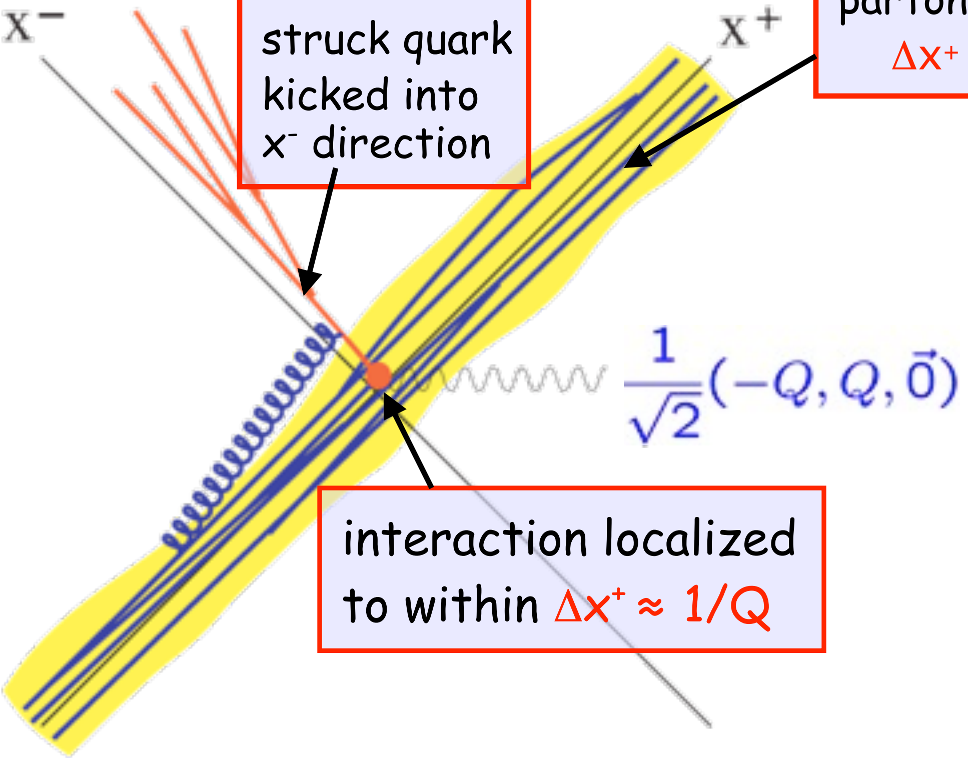

struck quark kicked into $x^-$ direction

interactions of partons dilated $\Delta x^+ \approx Q/m^2$

interaction localized to within $\Delta x^+ \approx 1/Q$

**upshot:**

- partons are free during the hard interaction
- lepton scatters off free partons incoherently
- convenient to introduce momentum fractions

$$0 < \xi_i \equiv p_i^+/p^+ < 1$$

# sum rules and isospin

for the quark distributions in a proton there are several sum rules to obey

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

momentum sum rule

quarks share proton momentum

$$\int_0^1 dx \left( f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left( f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

$$\int_0^1 dx \left( f_s^{(p)}(x) - f_{\bar{s}}^{(p)}(x) \right) = 0$$

flavor sum rules

conservation of quantum numbers

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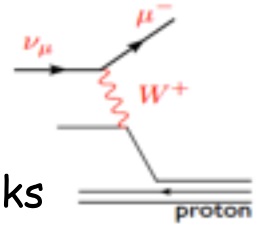
flavor sum rules

conservation of quantum numbers

isospin symmetry relates a neutron to a proton (just u and d interchanged)

$$F_2^n(x) = x\left(\frac{1}{9}d_n(x) + \frac{4}{9}u_n(x)\right) = x\left(\frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)\right)$$

- measuring both allows to determine $u^p$ and $d^p$ separately
- note: CC DIS couples to weak charges and separates quarks and antiquarks

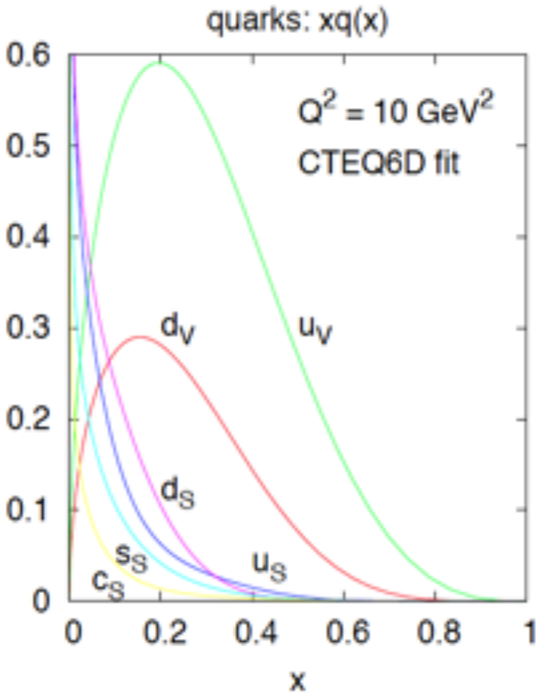


# momentum sum rule in the naïve parton model

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

| | |
|---|---|
| $u_v$ | 0.267 |
| $d_v$ | 0.111 |
| $u_s$ | 0.066 |
| $d_s$ | 0.053 |
| $s_s$ | 0.033 |
| $c_c$ | 0.016 |
| total | 0.546 |

half of the momentum is missing



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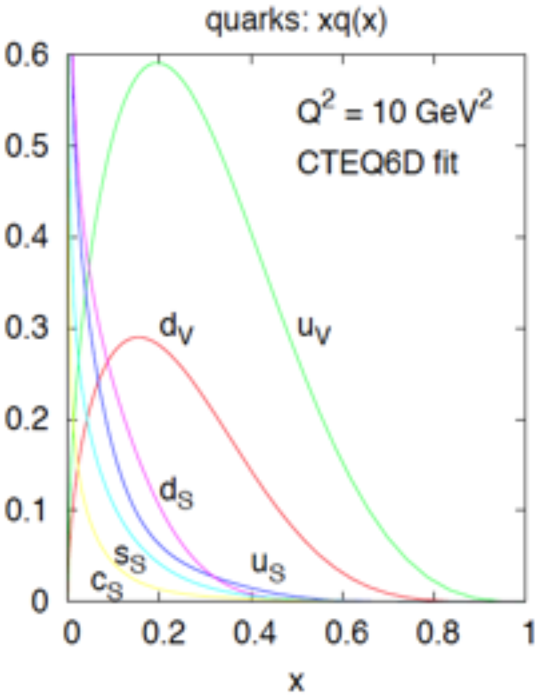

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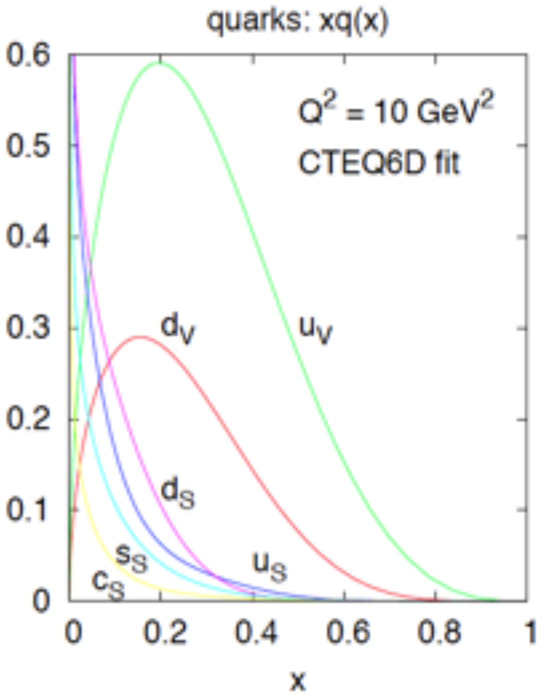
half of the momentum is missing

gluons !

but they don't carry electric/weak charge

how can they couple?



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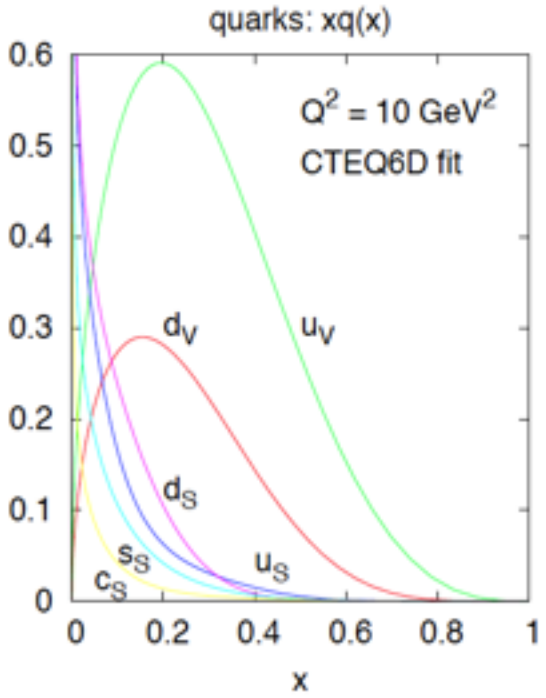

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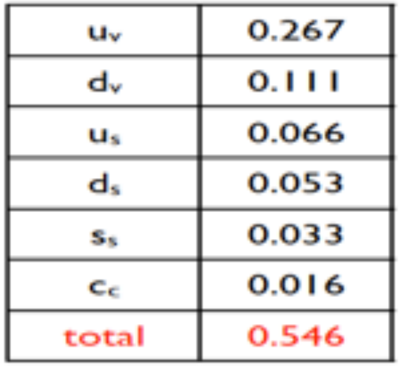
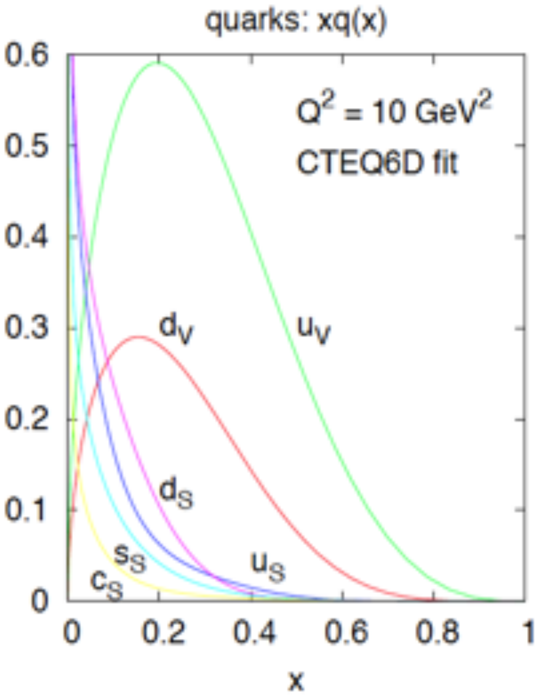
-> we need to discuss **QCD radiative corrections** to the naïve picture

# momentum sum rule in the naïve parton model

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$



| | |
|---|---|
| $u_v$ | 0.267 |
| $d_v$ | 0.111 |
| $u_s$ | 0.066 |
| $d_s$ | 0.053 |
| $s_s$ | 0.033 |
| $c_c$ | 0.016 |
| total | 0.546 |



half of the momentum is missing

gluons !

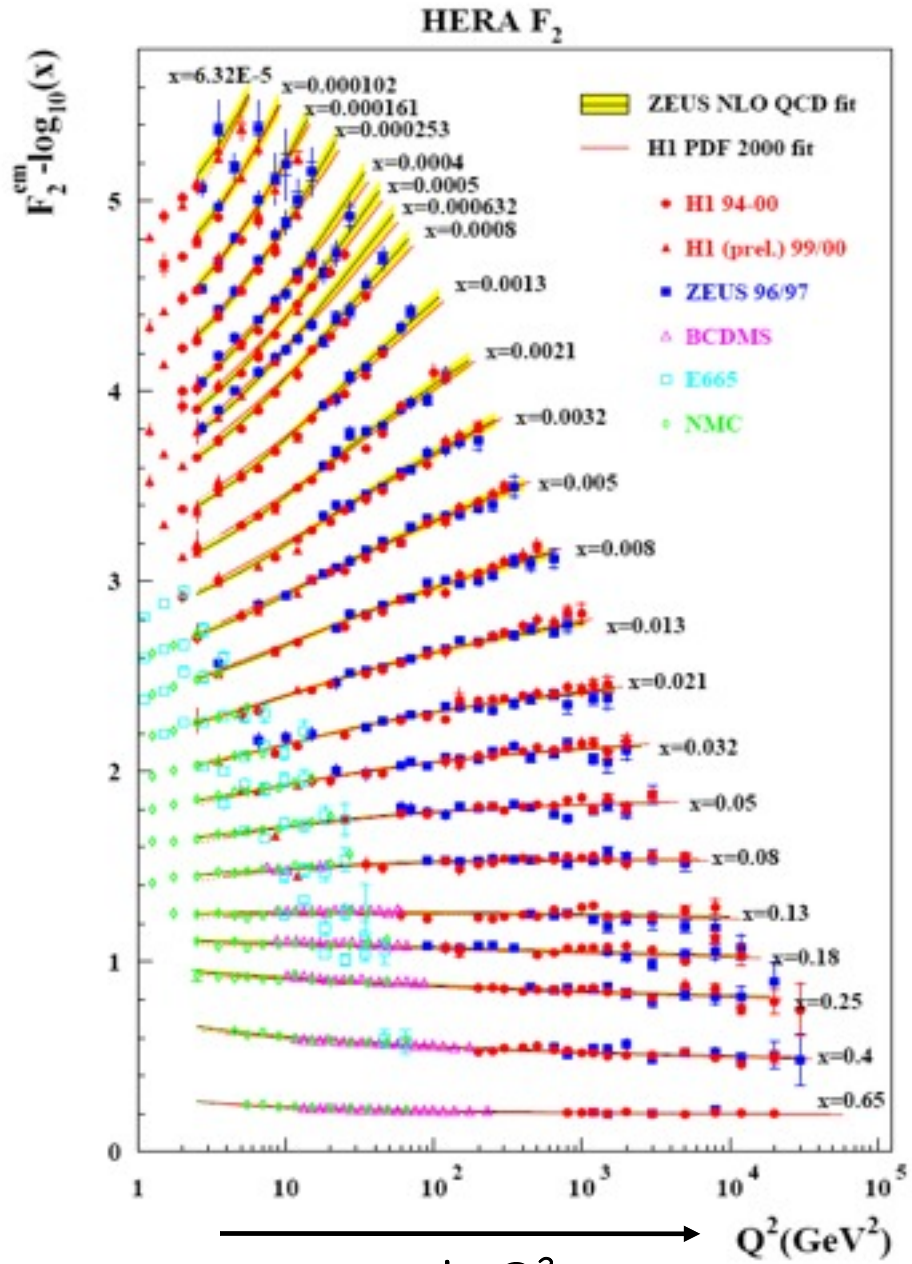
but they don't carry electric/weak charge

how can they couple?

-> we need to discuss QCD radiative corrections to the naïve picture

**gluons will enter the game and everything will become scale dependent**

# Naïve parton model vs. experiment



find strong scaling violations

scale $Q^2$

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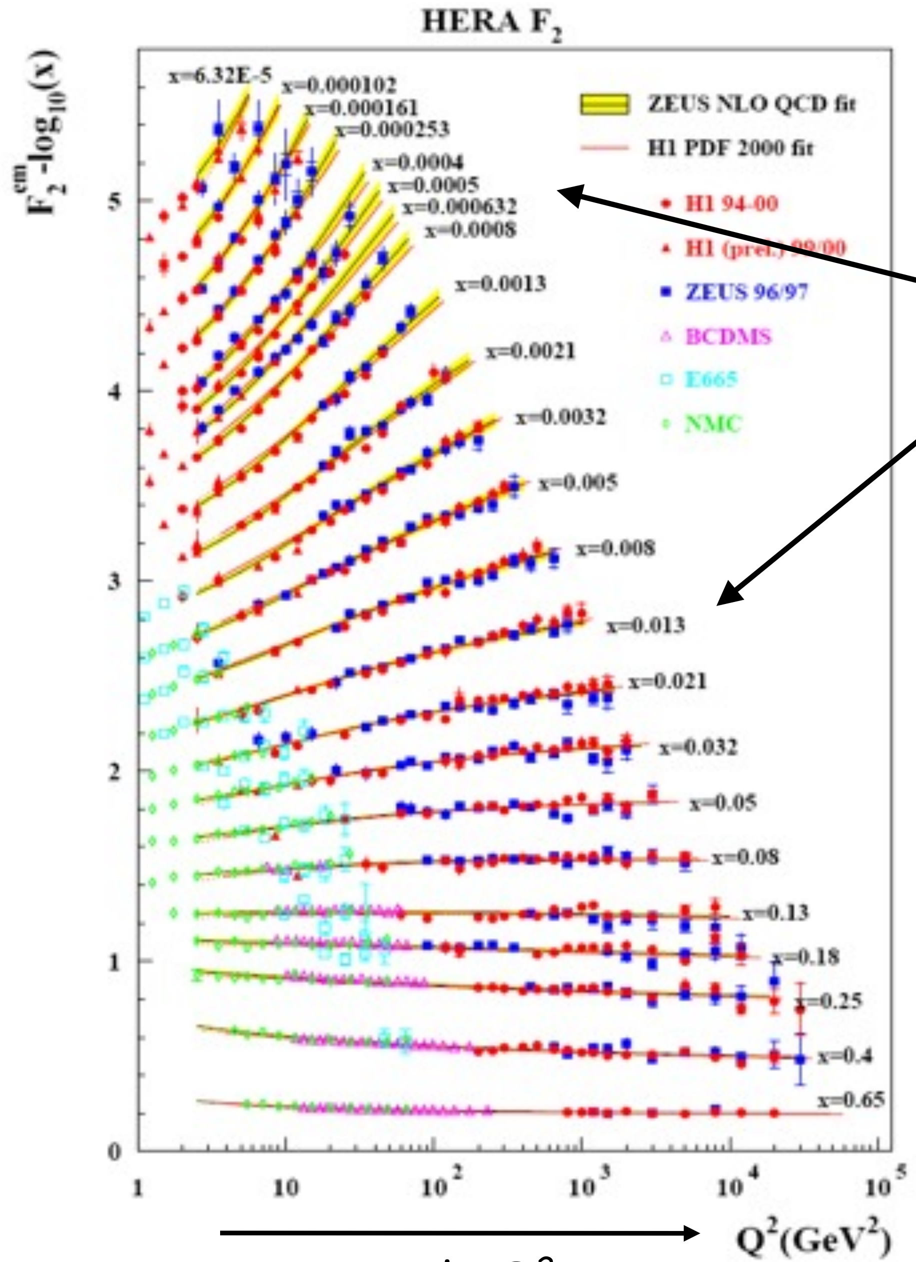

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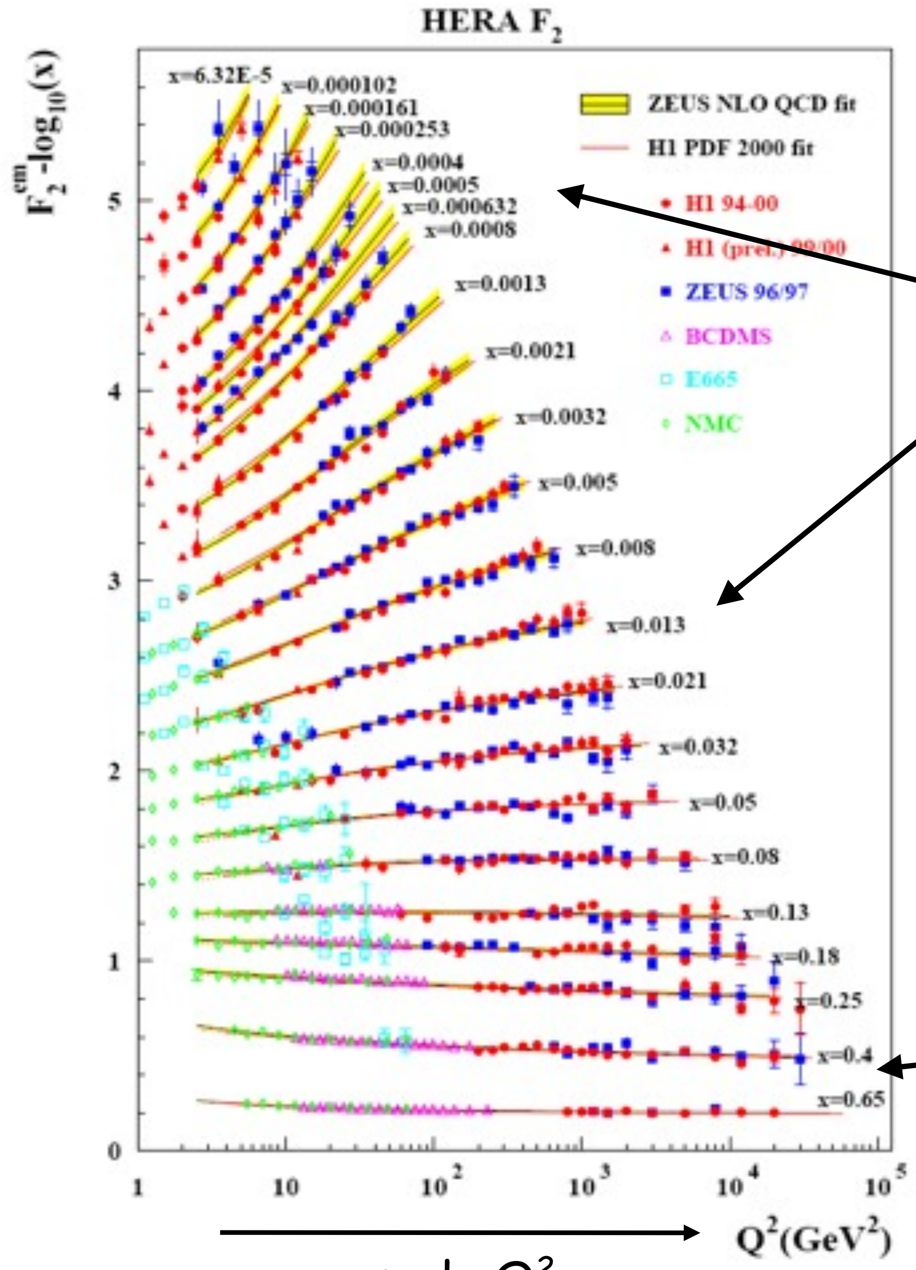

find strong scaling violations

significant rise at small x

decrease at high x

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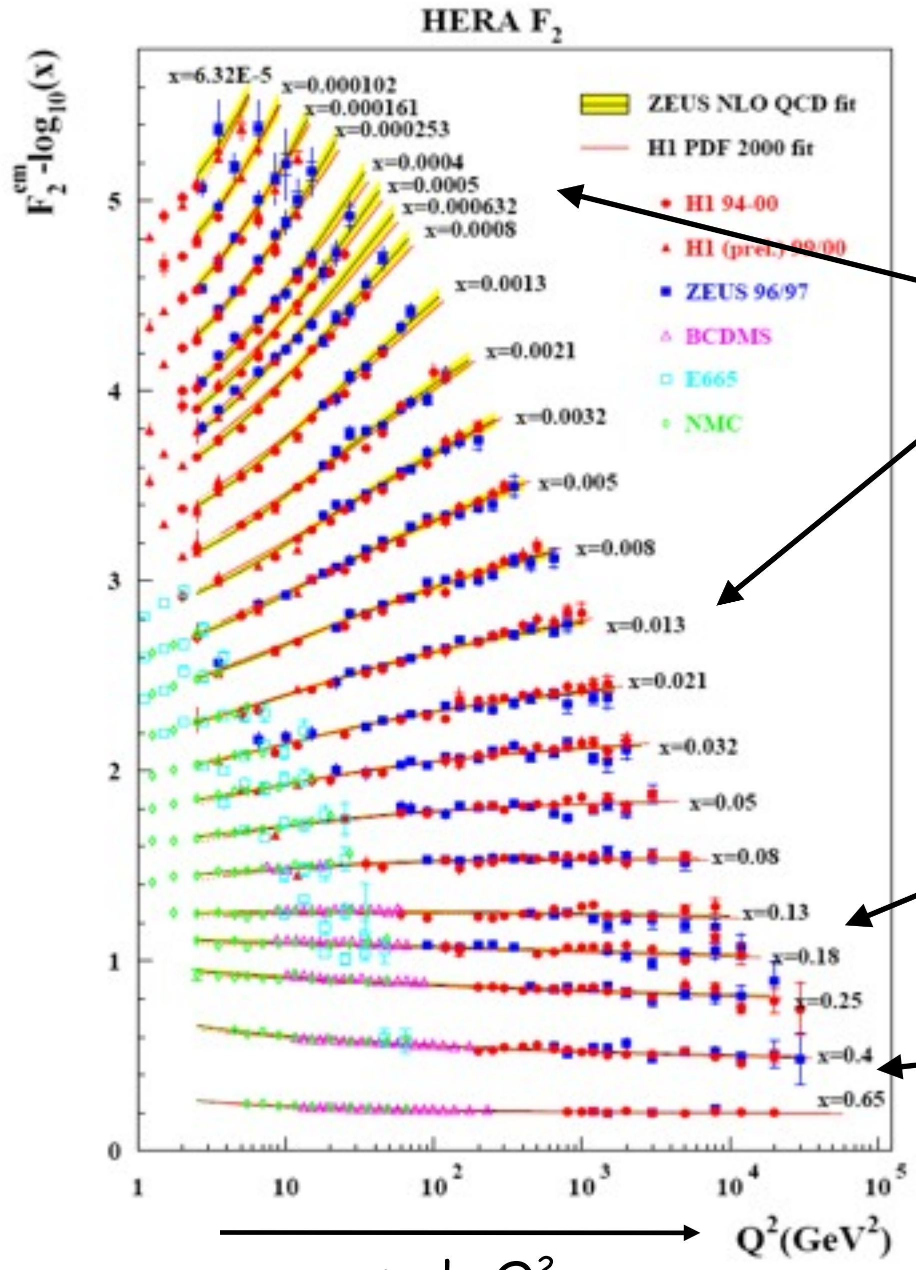


find strong scaling violations

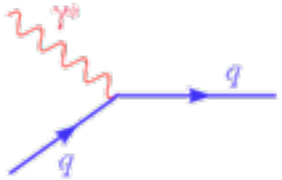
significant rise at small x

approximate scaling only around $x \simeq 0.15$

decrease at high x

scale $Q^2$

# DIS in the QCD improved parton model



we got a long way (parton model) without invoking QCD

now we have to study **QCD dynamics in DIS**

– this leads to similar problems already encountered in $e^+e^-$

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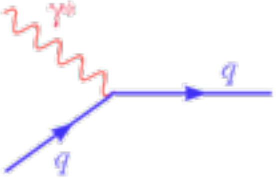

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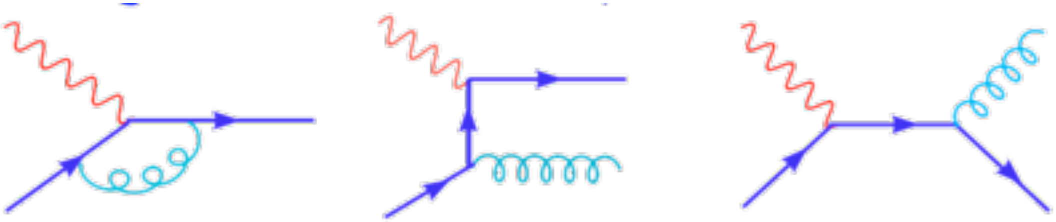
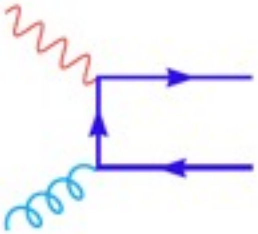


$\alpha_S$ corrections to the LO process



photon-gluon fusion

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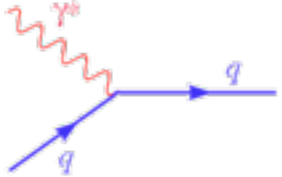

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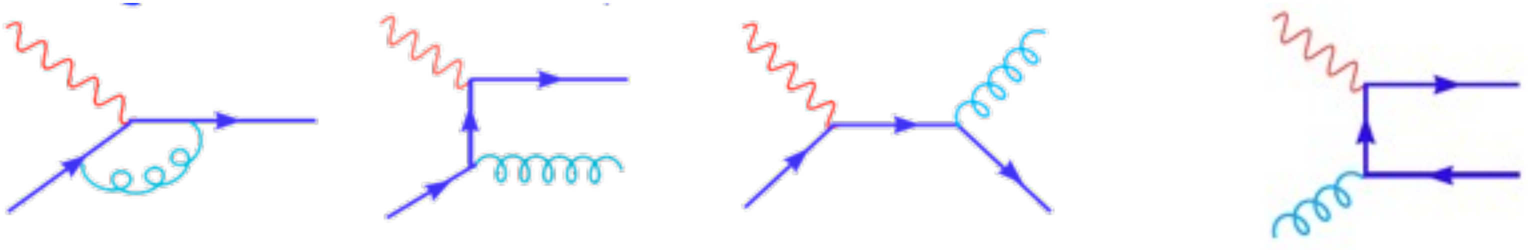


$\alpha_S$ corrections to the LO process

photon-gluon fusion

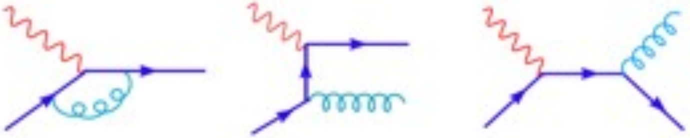
**caveat:** have to expect divergencies (recall 2nd part)
related to soft/collinear emission or from loops

we cannot calculate with infinities → introduce a "**regulator**"
and remove it in the end

# general structure of the O($\alpha_s$) corrections

using small (artificial) quark/gluon masses as regulator we obtain:

$$\left.\frac{d^2\hat{\sigma}}{dxdQ^2}\right|_{F_2} \equiv \hat{F}_2^q$$



$$= e_q^2 x\left[\delta(1-x)+\frac{\alpha_s(\mu_r)}{4\pi}\left[P_{qq}(x)\ln\frac{Q^2}{m_g^2}+C_2^q(x)\right]\right]$$

$$\left.\frac{d^2\hat{\sigma}}{dxdQ^2}\right|_{F_2} \equiv \hat{F}_2^g$$

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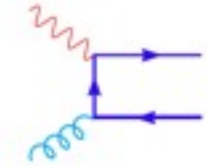

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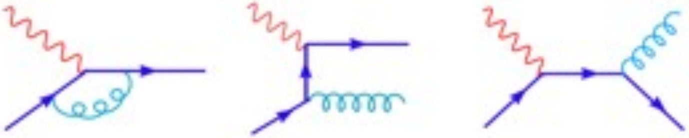
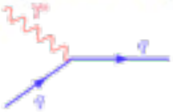


$$\begin{aligned}\left.\frac{d^2\hat{\sigma}}{dx\,dQ^2}\right|_{F_2} &\equiv \hat{F}_2^q \\ &= e_q^2 x\left[\overset{\text{LO}}{\delta(1-x)} + \frac{\alpha_s(\mu_r)}{4\pi}\left[P_{qq}(x)\ln\frac{Q^2}{m_g^2} + C_2^q(x)\right]\right]\end{aligned}$$



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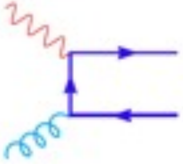

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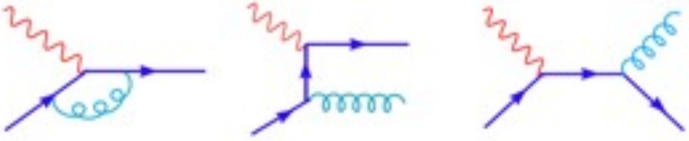
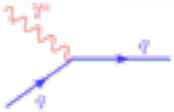


$$\left.\frac{d^2\hat{\sigma}}{dx\,dQ^2}\right|_{F_2} \equiv \hat{F}_2^q = e_q^2 x\left[\overset{\text{LO}}{\delta(1-x)} + \frac{\alpha_s(\mu_r)}{4\pi}\left[P_{qq}(x)\ln\frac{Q^2}{m_g^2} + C_2^q(x)\right]\right]$$



large logarithms
(collinear emission)

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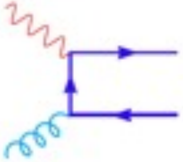

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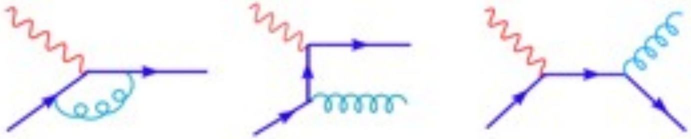

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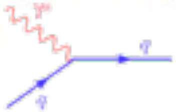

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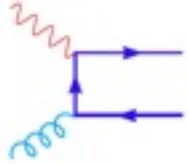

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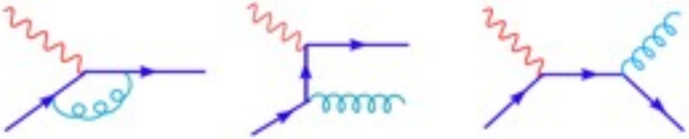

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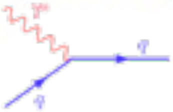

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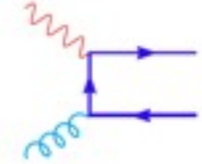

to see what happens to the logs we have to convolute our results with the PDFs

# factorization of collinear singularities

for the quark part we obtain:

$$F_2(x,Q^2) \;=\; x \sum_{a=q,\bar{q}} e_q^2 \Big[ f_{a,0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \\ f_{a,0}(x) \Big[ P_{qq}\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{m_g^2} + C_2^q\left(\frac{x}{\xi}\right) \Big]\Big]$$

similarly for the gluonic part

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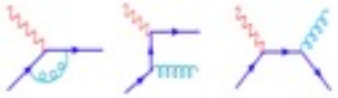
from 

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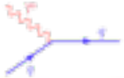

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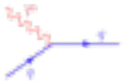

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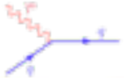

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**physical/renormalized densities**: not calculable in pQCD but **universal**

# general structure of a factorized cross section

putting everything together, keeping only terms up to $\alpha_s$:

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yet another scale: $\mu_r$

due to the renormalization
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choice of the factorization scheme

**this result is readily extended to hadron-hadron collisions**

# lesson: theorists are not afraid of infinities

ALRIGHT RUTH, I ABOUT GOT THIS ONE RENORMALIZED.

# universal PDFs → key to predictive power of pQCD

once PDFs are extracted from one set of experiments, e.g. DIS, we can use them to **predict cross sections** in, say, hadron-hadron collisions

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(closely linked to dim. regularization; used in all PDF fits)

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classic (but old-fashioned) definition of PDFs through their Mellin moments in Wilson-Zimmermann's operator product expansion (OPE)

Bardeen, Buras,
Duke, Muta

# PDFs as bi-local operators

Curci, Furmanski,
Petronzio; Collins, Soper
see, e.g., D. Soper,
hep-lat/9609018

**more physical formulation** in Bjorken-x space:
matrix elements of bi-local operators on the light-cone

for quarks: (similar for gluons; easy to include spin $\gamma^+ \to \gamma^+\gamma_5$)

$$\boxed{f_a(\xi,\mu_f) = \frac{1}{2}\int \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} \langle p|\overline{\Psi}_a(0,y^-,\vec{0})\gamma^+ \mathcal{F}\Psi_a(0)|p\rangle_{\overline{\mathrm{MS}}}}$$

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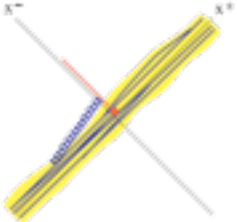
Foundations of Perturbative QCD

JOHN COLLINS

$$\mathcal{F} = \mathcal{P}\exp\left(-ig\int_0^{y^-} dz^- A_c^+(0,z^-,\vec{0})T_c\right)$$



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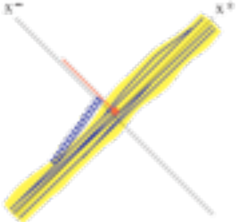

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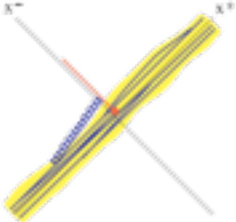
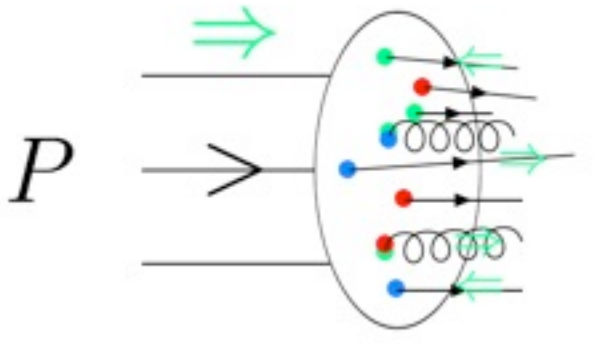


crucial role for a special class of "**transverse-momentum dep. PDFs**" describing phenomena with transverse polarization (**"Sivers function"**, ...)

- interpretation as number operator only in "**$A^+= 0$ gauge**"
- turn into local operators (→ lattice QCD) if taking moments $\int_0^1 d\xi\, \xi^n$

# pictorial representation of PDFs

suppose we could take a snapshot of a nucleon with positive helicity



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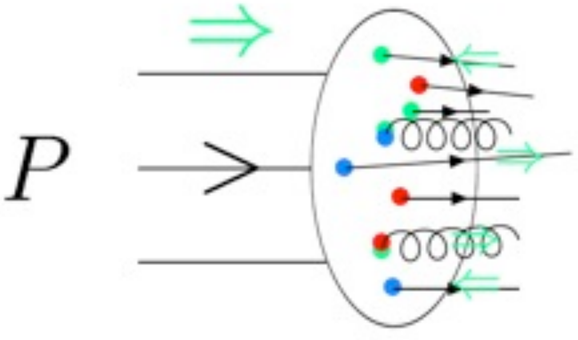

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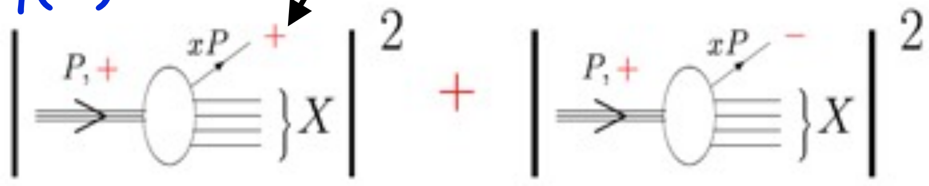
helicity



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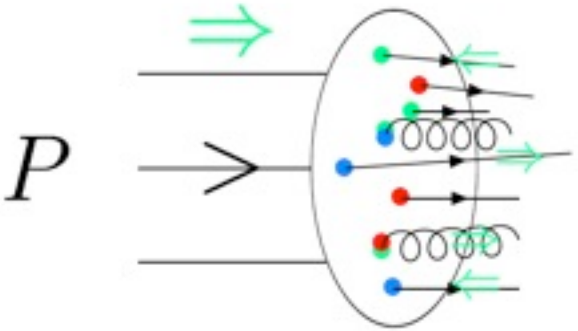

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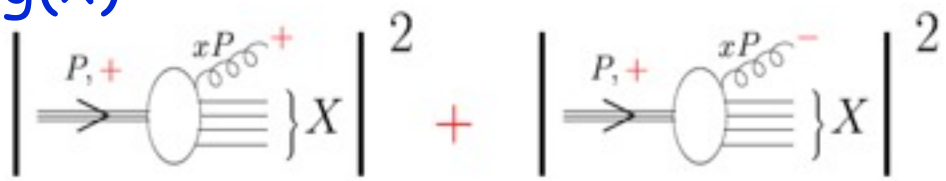

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**unpolarized PDFs**

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**helicity-dep. PDFs**

→ spin of the nucleon

# towards renormalization group equations

**so far:** infinities related to long-time/distance physics (soft/collinear emissions)

these singularities cancel for infrared safe observables
or can be systematically removed (factorization) by "hiding" them
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we can insert perturbative corrections
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loop momenta can be very large (=infinite)
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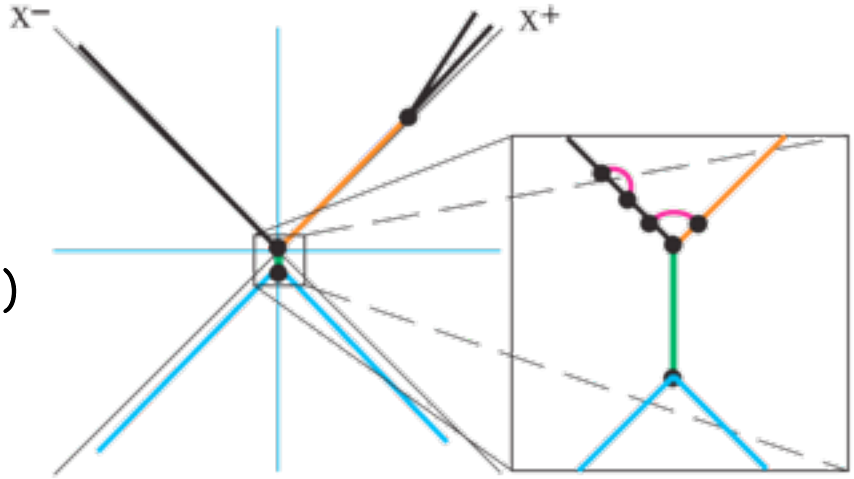

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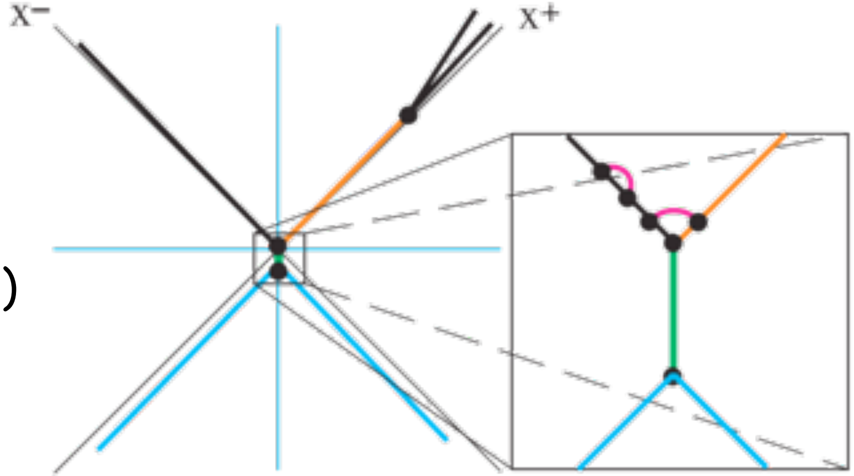

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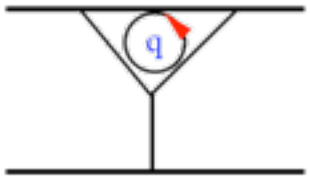
again, we need a suitable regulator for
divergent loop integrations:

**UV cut-off** vs. **dim. regularization**

intuitive;
not beyond NLO

involved;
works to all orders

$$\int_0^\infty d^4q$$



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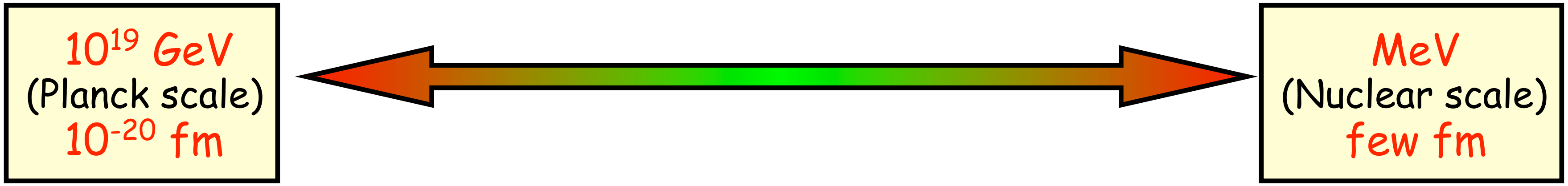

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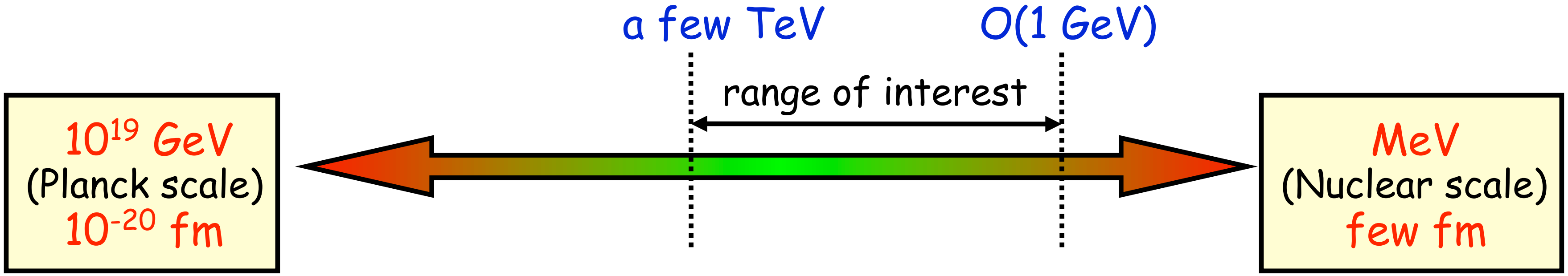

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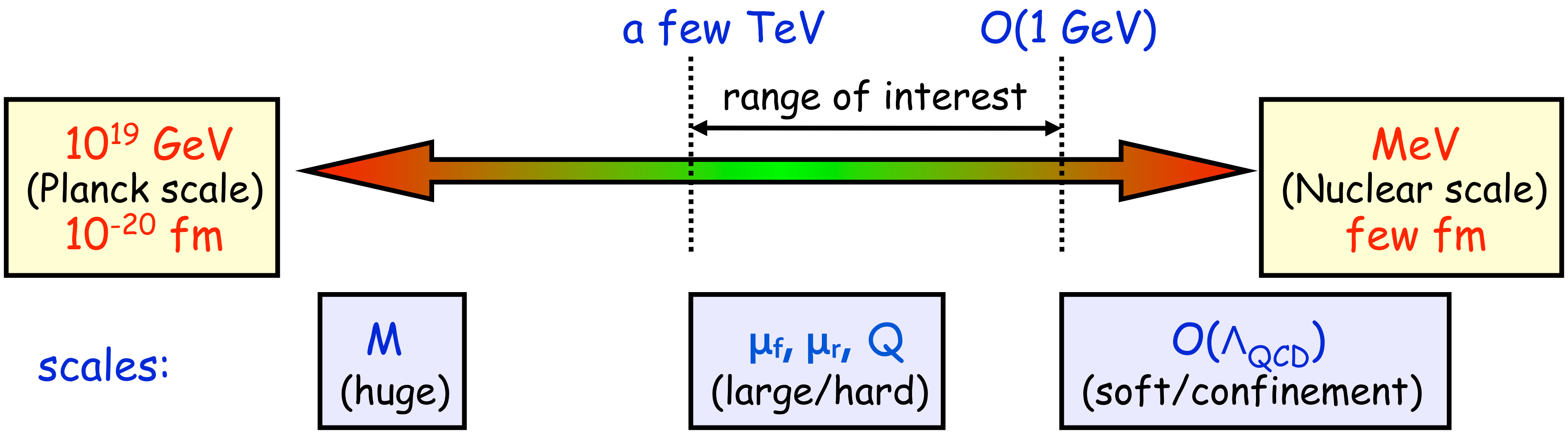

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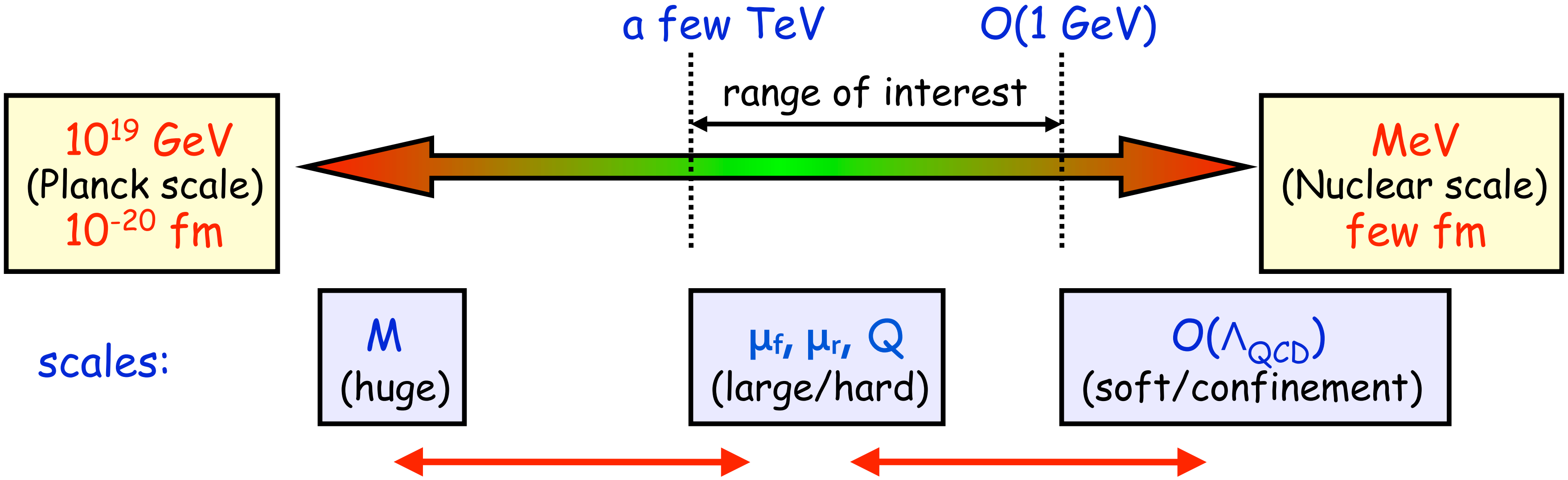

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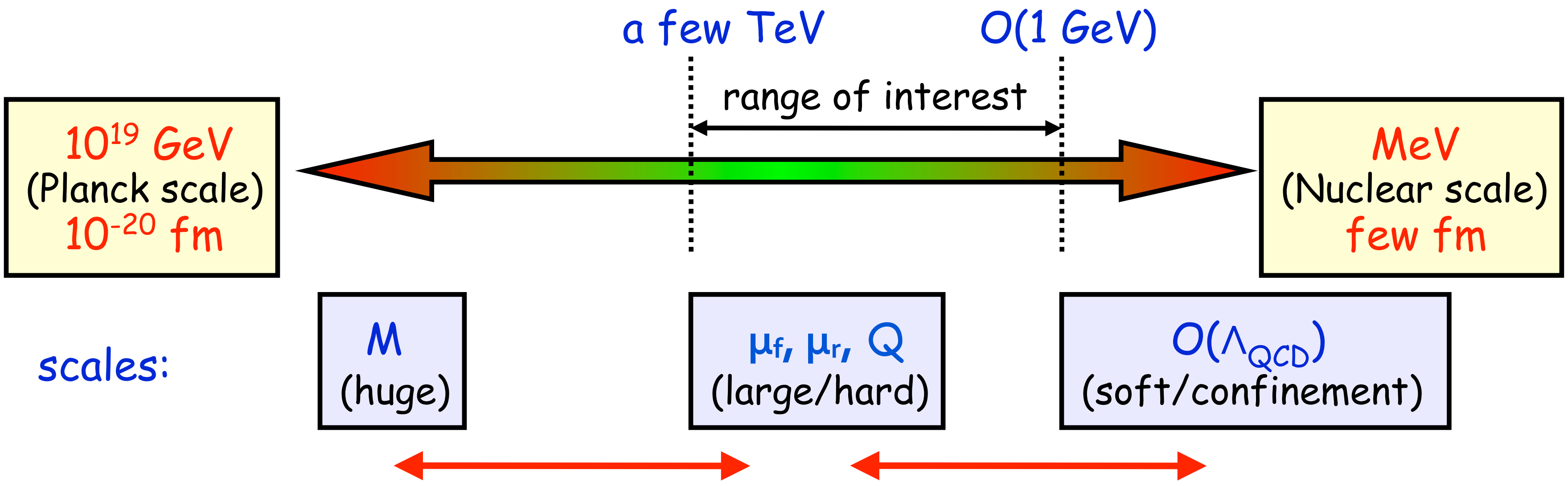

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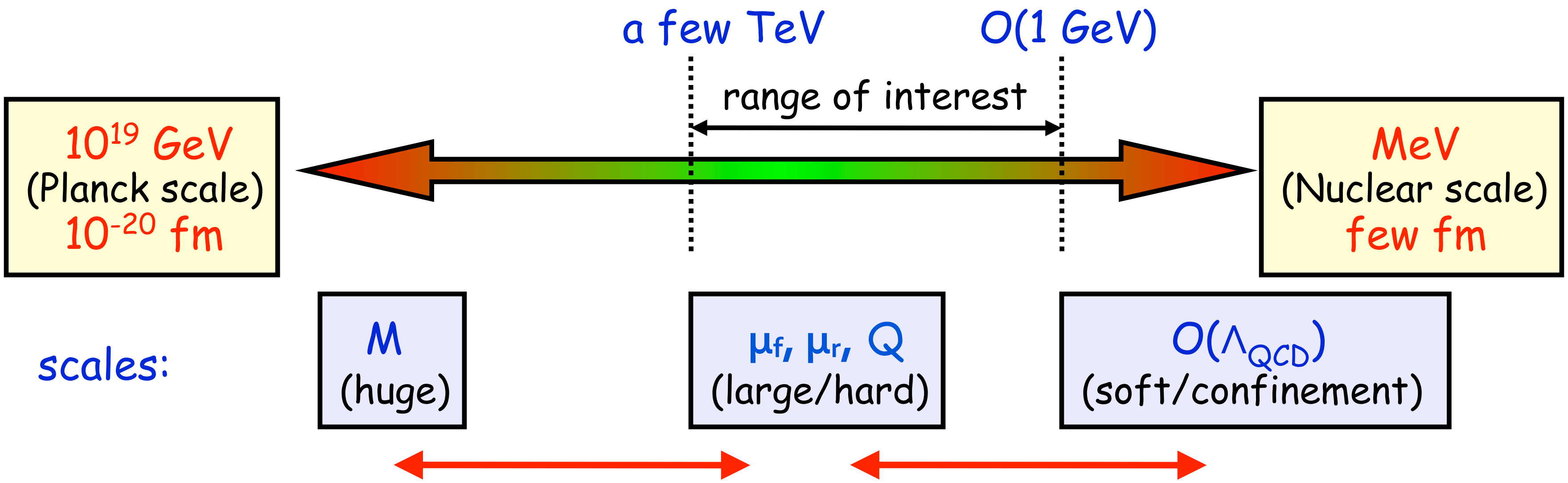

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both scale parameters $\mu_f$ and $\mu_r$ are not intrinsic to QCD

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all we need is a reference measurement at some scale $\mu_0$

# scale evolution of $\alpha_s$ and parton densities

**simplest example of RGE:** running coupling $\alpha_s$ derived from $\frac{d\sigma}{d \ln \mu_r} = 0$

→ recall part II $$\frac{da_s}{d \ln \mu^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + \ldots \qquad a_s \equiv \frac{\alpha_s}{4\pi}$$

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simplified example:
$F_2$ for one quark flavor

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$$\int_0^1 dx x^{n-1} \int_0^1 dy \int_0^1 dz\, \delta(x - zy) f(y) g(z) = f(n)\, g(n)$$

# simplest example of DGLAP evolution

**Dokshitzer; Gribov, Lipatov; Altarelli, Parisi**

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↔ $$\frac{dq(n,\mu_f)}{d\ln\mu_f}\hat{F}_2(n,\frac{Q}{\mu_f})+q(n,\mu_f)\frac{d\hat{F}_2(n,\frac{Q}{\mu_f})}{d\ln\mu_f}=0$$

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splitting function

**DGLAP** evolution equation

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splitting function

**DGLAP evolution equation**

solve it

$$\boxed{q(n,\mu_f) = q(n,\mu_0)\exp\left[\frac{\alpha_s}{2\pi}P_{qq}(n)\ln\left(\frac{\mu_f}{\mu_0}\right)\right]}$$

disclaimer: kept $\alpha_s$ constant for simplicity

# simplest example of DGLAP evolution

Dokshitzer; Gribov, Lipatov; Altarelli, Parisi

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→ **once we know the PDFs at a scale $\mu_0$ we can predict them at $\mu > \mu_0$**

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**physical interpretation of the evolution eqs.:**

RGE resums collinear emissions to all orders

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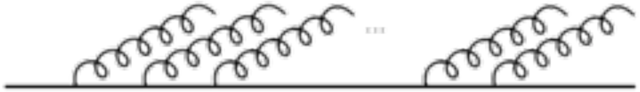

$$\exp[\ldots] = 1 + \frac{\alpha_s}{2\pi} P_{qq}(n) \ln \frac{\mu_f}{\mu_0} + \frac{1}{2}\left[\frac{\alpha_s}{2\pi} P_{qq}(n) \ln \frac{\mu_f}{\mu_0}\right]^2 + \ldots$$

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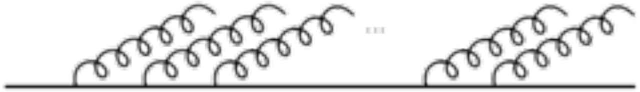

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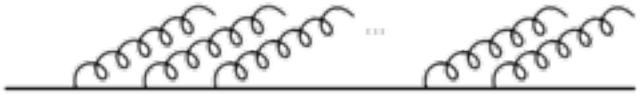

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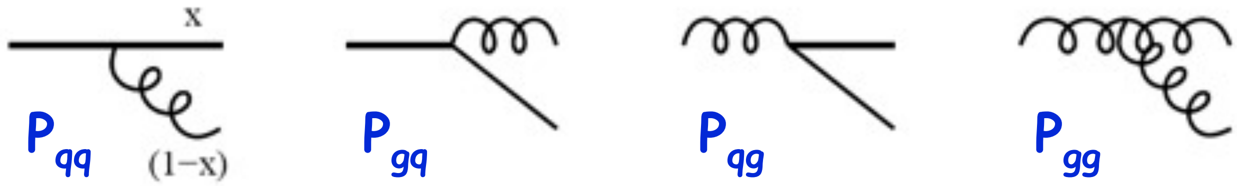
- the **splitting functions** $P_{ij}(n)$ **or** $P_{ij}(x)$ multiplying the log's are universal and calculable in pQCD order by order in $\alpha_s$
- the physical meaning of the splitting functions is easy:

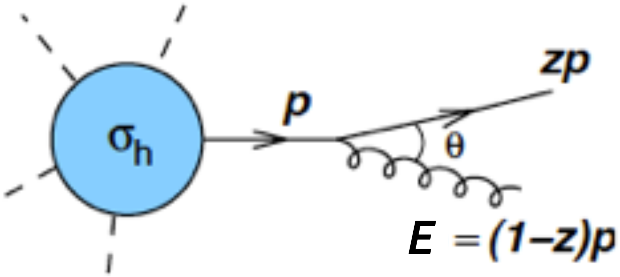
$P_{ij}(x)$ : probability that a parton j splits collinearly into a parton i (and something) carrying a momentum fraction x



# factorization recap: final-state vs initial-state

recall what we learned for **final-state radiation**

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta^2}{\theta^2}$$



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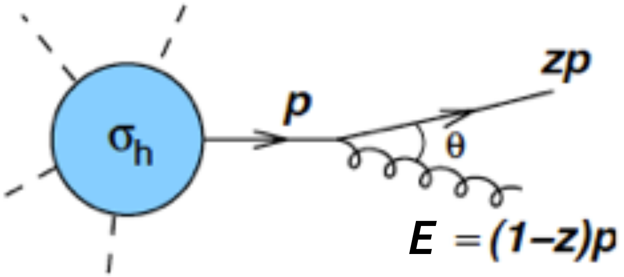

and rewrite in terms of new variable $k_T$

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

where we have used

$$\mathbf{E} = (\mathbf{1}-\mathbf{z})\mathbf{p}$$
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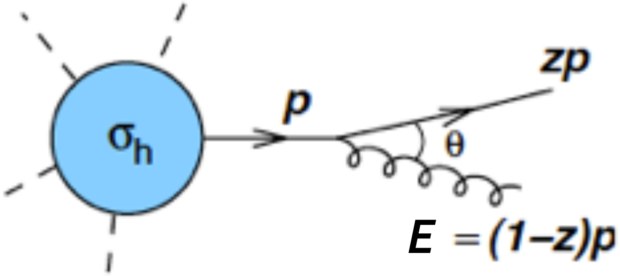

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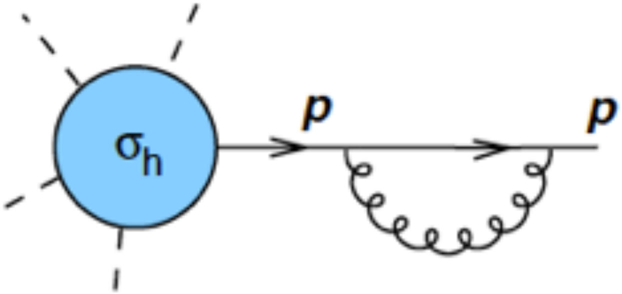
**KLN:** if we avoid distinguishing quark and collinear quark-gluon final-states (like for **jets**) divergencies cancel against virtual corrections



$$\sigma_{h+V} \simeq -\sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

# factorization recap: initial-state peculiarities

**initial-state radiation:** crucial difference – hard scattering happens after splitting

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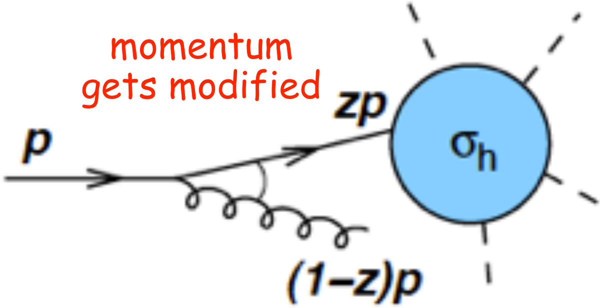

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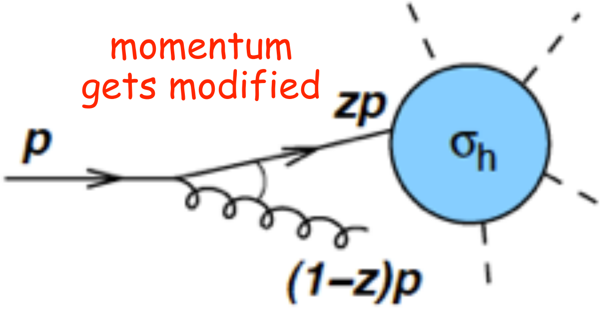
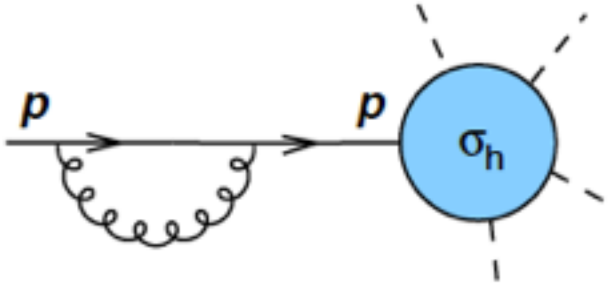

momentum gets modified

but for the virtual piece the momentum is unchanged



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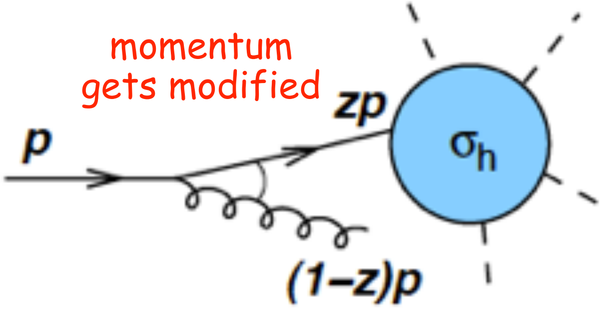
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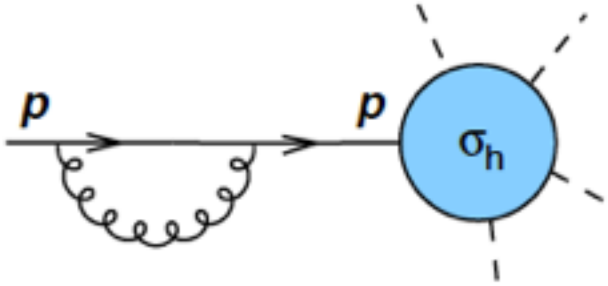

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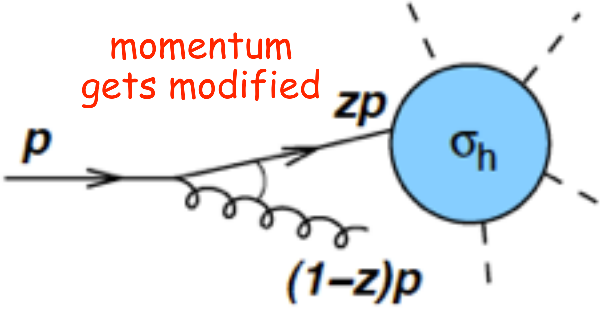
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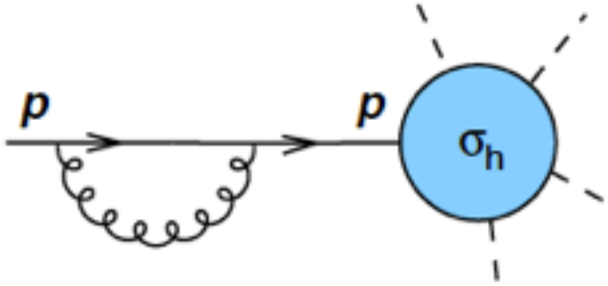

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leads to uncanceled collinear singularity

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$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_0^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{infinite}} \underbrace{\int \frac{dz}{1-z}[\sigma_h(zp) - \sigma_h(p)]}_{\text{finite}}$$

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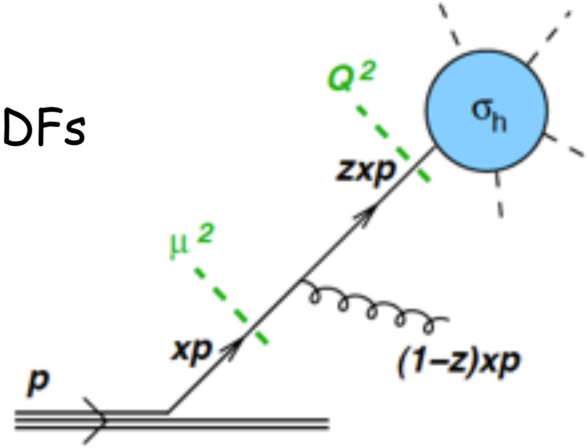
**factorization = collinear "cut-off"**

- absorb divergent small $k_T$ region in non-perturbative PDFs

$$\sigma_1 \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_{\mu^2}^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{finite (large)}} \underbrace{\int \frac{dx\,dz}{1-z}[\sigma_h(zxp) - \sigma_h(xp)]\, q(x,\mu^2)}_{\text{finite}}$$



# anatomy of splitting functions

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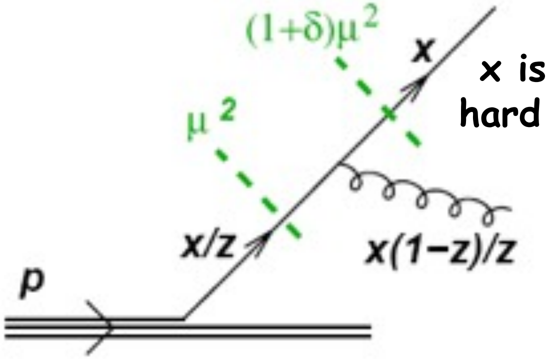

**x is fixed by hard scattering**

**real emission**
"something happens"

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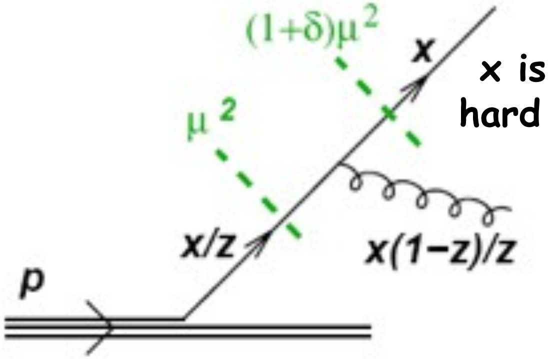

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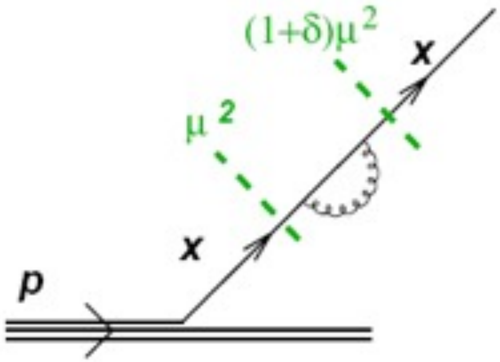

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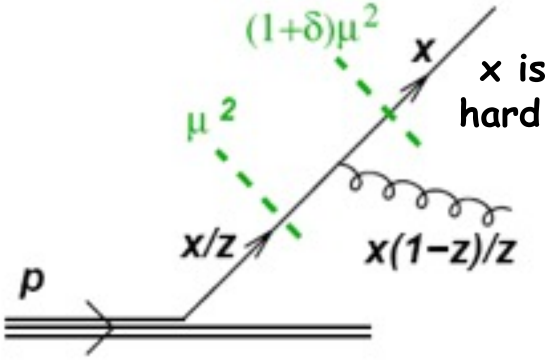

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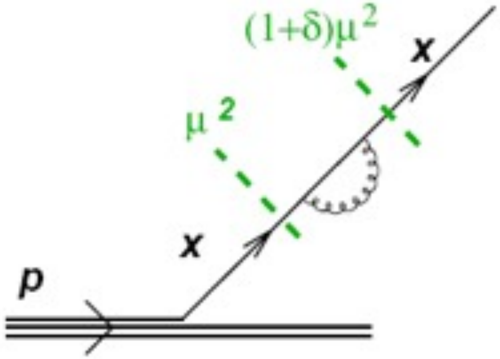

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**combine !**

$$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\underbrace{\int_x^1 dz P_{qq}(z)\frac{q(x/z,\mu^2)}{z}}_{P_{qq}\otimes q} \qquad P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

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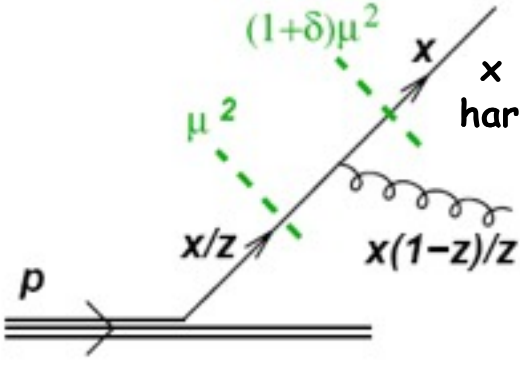

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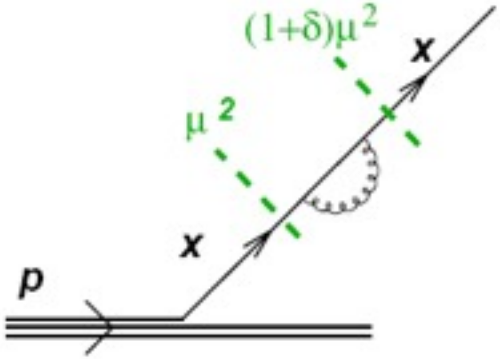

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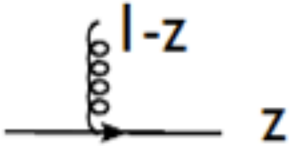
involves “plus distribution” $$\int_0^1 dz\,[g(z)]_+ f(z) \equiv \int_0^1 dz\, g(z)\,[f(z)-f(1)]$$

condition: f(z) sufficiently smooth for z→ 1

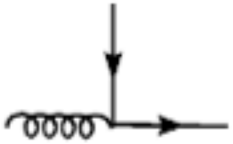
# properties of LO splitting functions

in general, quarks and gluons can split into quarks and gluons -> 4 functions

$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F \left[ \frac{\mathbf{1+z^2}}{\mathbf{(1-z)}_+} + \frac{3}{2}\delta(1-z) \right]$$



$$P_{qg}^{(0)} = P_{\bar{q}g}^{(0)} = T_R \left( z^2 + (1-z) \right)$$



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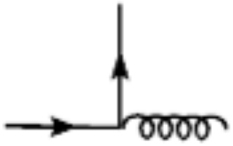
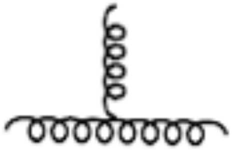


$$P_{gg}^{(0)} = 2C_A \left[ z\left(\frac{1}{1-z}\right)_+ + \frac{1-z}{z} + z(1-z) + b_0\delta(1-z) \right]$$



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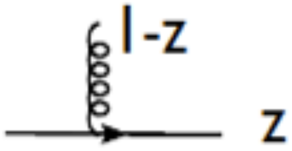

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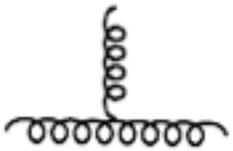

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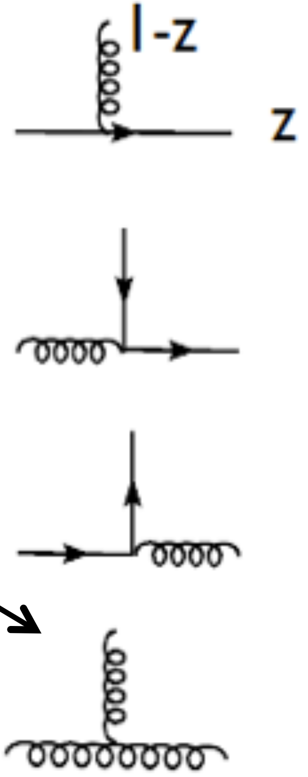

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symmetric under
z -> (1-z)
except virtuals

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# reaching for precision

$$
\begin{aligned}
P^{(0)}_{\mathrm{ns}}(x) &= C_F\left(2p_{\mathrm{qq}}(x)+3\delta(1-x)\right)\\
P^{(0)}_{\mathrm{ps}}(x) &= 0\\
P^{(0)}_{\mathrm{qg}}(x) &= 2n_f\,p_{\mathrm{qg}}(x)\\
P^{(0)}_{\mathrm{gq}}(x) &= 2C_F\,p_{\mathrm{gq}}(x)\\
P^{(0)}_{\mathrm{gg}}(x) &= C_A\left(4p_{\mathrm{gg}}(x)+\frac{11}{3}\delta(1-x)\right)-\frac{2}{3}n_f\,\delta(1-x)
\end{aligned}
$$

**LO: 1973**

# reaching for precision

$$
\begin{aligned}
P^{(0)}_{\mathrm{ns}}(x) &= C_F\left(2p_{\mathrm{qq}}(x) + 3\delta(1-x)\right)\\
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P^{(0)}_{\mathrm{qg}}(x) &= 2n_f\, p_{\mathrm{qg}}(x)\\
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P^{(0)}_{\mathrm{gg}}(x) &= C_A\left(4p_{\mathrm{gg}}(x) + \frac{11}{3}\delta(1-x)\right) - \frac{2}{3}n_f\,\delta(1-x)
\end{aligned}
$$

**LO: 1973**

$$
\begin{aligned}
P^{(1)+}_{\mathrm{ns}}(x) = {}& 4C_AC_F\Big(p_{\mathrm{qq}}(x)\Big[\frac{67}{18} - \zeta_2 + \frac{11}{6}\mathrm{H}_0 + \mathrm{H}_{0,0}\Big] + p_{\mathrm{qq}}(-x)\Big[\zeta_2 + 2\mathrm{H}_{-1,0} - \mathrm{H}_{0,0}\Big]\\
&+ \frac{14}{3}(1-x) + \delta(1-x)\Big[\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3\Big]\Big) - 4C_Fn_f\Big(p_{\mathrm{qq}}(x)\Big[\frac{5}{9} + \frac{1}{3}\mathrm{H}_0\Big] + \frac{2}{3}(1-x)\\
&+ \delta(1-x)\Big[\frac{1}{12} + \frac{2}{3}\zeta_2\Big]\Big) + 4C_F{}^2\Big(2p_{\mathrm{qq}}(x)\Big[\mathrm{H}_{1,0} - \frac{3}{4}\mathrm{H}_0 + \mathrm{H}_2\Big] - 2p_{\mathrm{qq}}(-x)\Big[\zeta_2 + 2\mathrm{H}_{-1,0}\\
&- \mathrm{H}_{0,0}\Big] - (1-x)\Big[1 - \frac{3}{2}\mathrm{H}_0\Big] - \mathrm{H}_0 - (1+x)\mathrm{H}_{0,0} + \delta(1-x)\Big[\frac{3}{8} - 3\zeta_2 + 6\zeta_3\Big]\Big)\\
P^{(1)-}_{\mathrm{ns}}(x) = {}& P^{(1)+}_{\mathrm{ns}}(x) + 16C_F\Big(C_F - \frac{C_A}{2}\Big)\Big(p_{\mathrm{qq}}(-x)\Big[\zeta_2 + 2\mathrm{H}_{-1,0} - \mathrm{H}_{0,0}\Big] - 2(1-x)\\
&- (1+x)\mathrm{H}_0\Big)
\end{aligned}
$$

$$
P^{(1)}_{\mathrm{ps}}(x) = 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4\mathrm{H}_0 + x^2\Big[\frac{8}{3}\mathrm{H}_0 - \frac{56}{9}\Big] + (1+x)\Big[5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\Big]\Big)
$$

$$
\begin{aligned}
P^{(1)}_{\mathrm{qg}}(x) = {}& 4C_An_f\Big(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\mathrm{qg}}(-x)\mathrm{H}_{-1,0} - 2p_{\mathrm{qg}}(x)\mathrm{H}_{1,1} + x^2\Big[\frac{44}{3}\mathrm{H}_0 - \frac{218}{9}\Big]\\
&+ 4(1-x)\Big[\mathrm{H}_{0,0} - 2\mathrm{H}_0 + x\mathrm{H}_1\Big] - 4\zeta_2x - 6\mathrm{H}_{0,0} + 9\mathrm{H}_0\Big) + 4C_Fn_f\Big(2p_{\mathrm{qg}}(x)\Big[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2\\
&- \zeta_2\Big] + 4x^2\Big[\mathrm{H}_0 + \mathrm{H}_{0,0} + \frac{5}{2}\Big] + 2(1-x)\Big[\mathrm{H}_0 + \mathrm{H}_{0,0} - 2x\mathrm{H}_1 + \frac{29}{4}\Big] - \frac{15}{2} - \mathrm{H}_{0,0} - \frac{1}{2}\mathrm{H}_0\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\mathrm{gq}}(x) = {}& 4C_AC_F\Big(\frac{1}{x} + 2p_{\mathrm{gq}}(x)\Big[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2 - \frac{11}{6}\mathrm{H}_1\Big] - x^2\Big[\frac{8}{3}\mathrm{H}_0 - \frac{44}{9}\Big] + 4\zeta_2 - 2\\
&- 7\mathrm{H}_0 + 2\mathrm{H}_{0,0} - 2\mathrm{H}_1x + (1+x)\Big[2\mathrm{H}_{0,0} - 5\mathrm{H}_0 + \frac{37}{9}\Big] - 2p_{\mathrm{gq}}(-x)\mathrm{H}_{-1,0}\Big) - 4C_Fn_f\Big(\frac{2}{3}x\\
&- p_{\mathrm{gq}}(x)\Big[\frac{2}{3}\mathrm{H}_1 - \frac{10}{9}\Big]\Big) + 4C_F{}^2\Big(p_{\mathrm{gq}}(x)\Big[3\mathrm{H}_1 - 2\mathrm{H}_{1,1}\Big] + (1+x)\Big[\mathrm{H}_{0,0} - \frac{7}{2} + \frac{7}{2}\mathrm{H}_0\Big] - 3\mathrm{H}_{0,0}\\
&+ 1 - \frac{3}{2}\mathrm{H}_0 + 2\mathrm{H}_1x\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\mathrm{gg}}(x) = {}& 4C_An_f\Big(1 - x - \frac{10}{9}p_{\mathrm{gg}}(x) - \frac{13}{9}\Big(\frac{1}{x} - x^2\Big) - \frac{2}{3}(1+x)\mathrm{H}_0 - \frac{2}{3}\delta(1-x)\Big) + 4C_A{}^2\Big(27\\
&+ (1+x)\Big[\frac{11}{3}\mathrm{H}_0 + 8\mathrm{H}_{0,0} - \frac{27}{2}\Big] + 2p_{\mathrm{gg}}(-x)\Big[\mathrm{H}_{0,0} - 2\mathrm{H}_{-1,0} - \zeta_2\Big] - \frac{67}{9}\Big(\frac{1}{x} - x^2\Big) - 12\mathrm{H}_0\\
&- \frac{44}{3}x^2\mathrm{H}_0 + 2p_{\mathrm{gg}}(x)\Big[\frac{67}{18} - \zeta_2 + \mathrm{H}_{0,0} + 2\mathrm{H}_{1,0} + 2\mathrm{H}_2\Big] + \delta(1-x)\Big[\frac{8}{3} + 3\zeta_3\Big]\Big) + 4C_Fn_f\Big(2\mathrm{H}_0\\
&+ \frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\Big[4 - 5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\Big] - \frac{1}{2}\delta(1 -
\end{aligned}
$$

Curci, Furmanski, Petronzio;
Floratos et al., …

**NLO: 1980**

# $P_{ij}$ @ NNLO: a landmark calculation

10000 diagrams, $10^5$ integrals, 10 man years, and several CPU years later:

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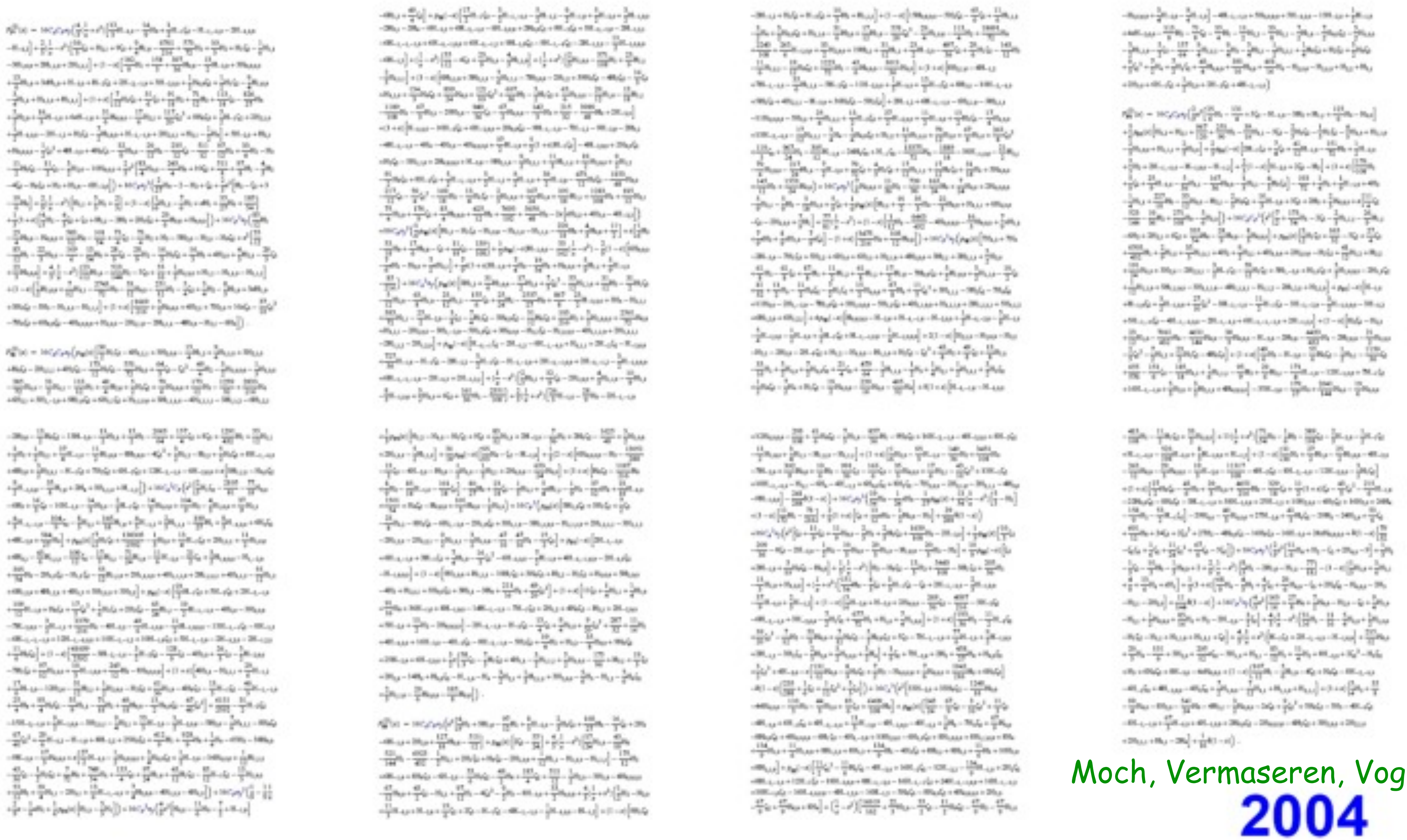

Moch, Vermaseren, Vogt

**2004**

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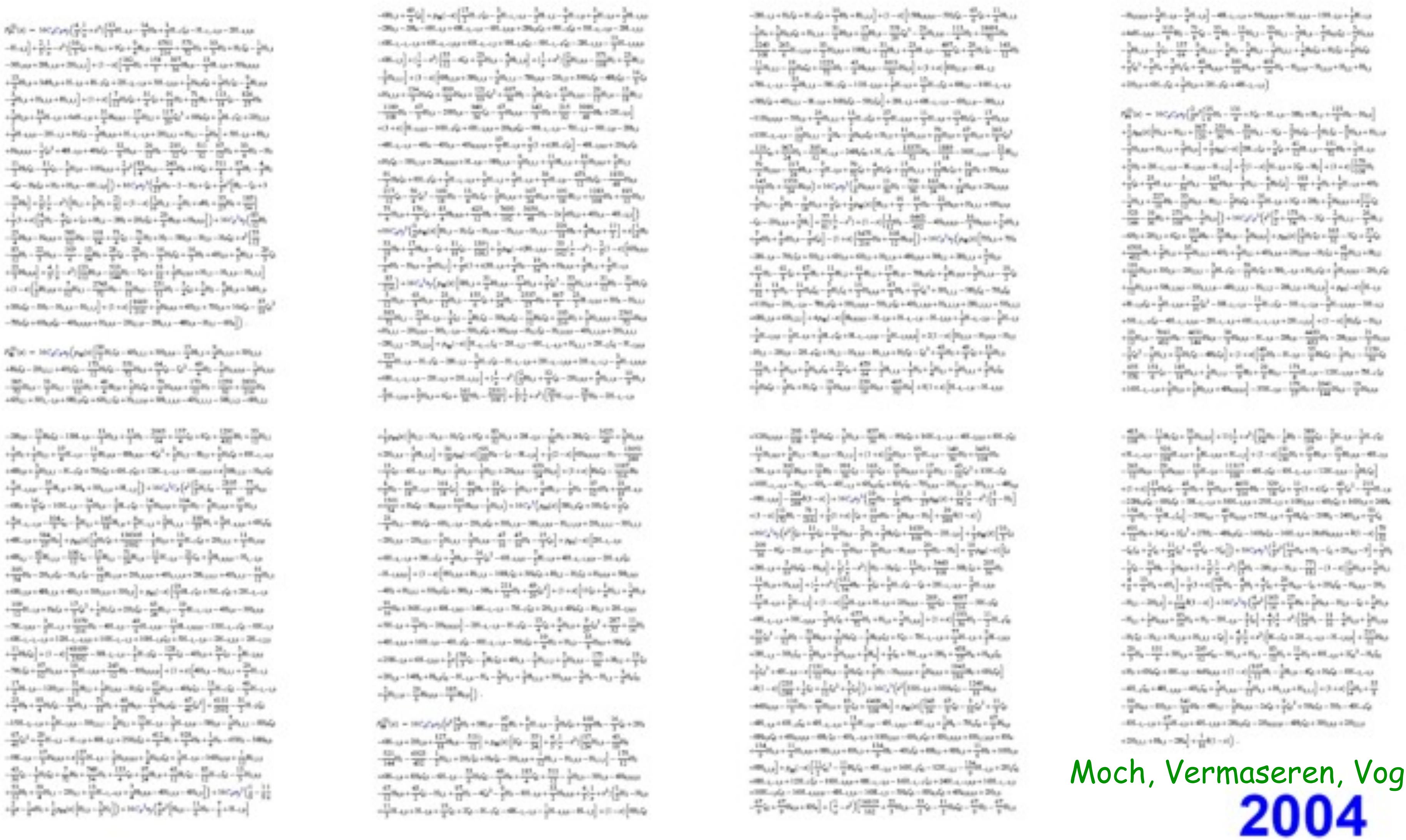

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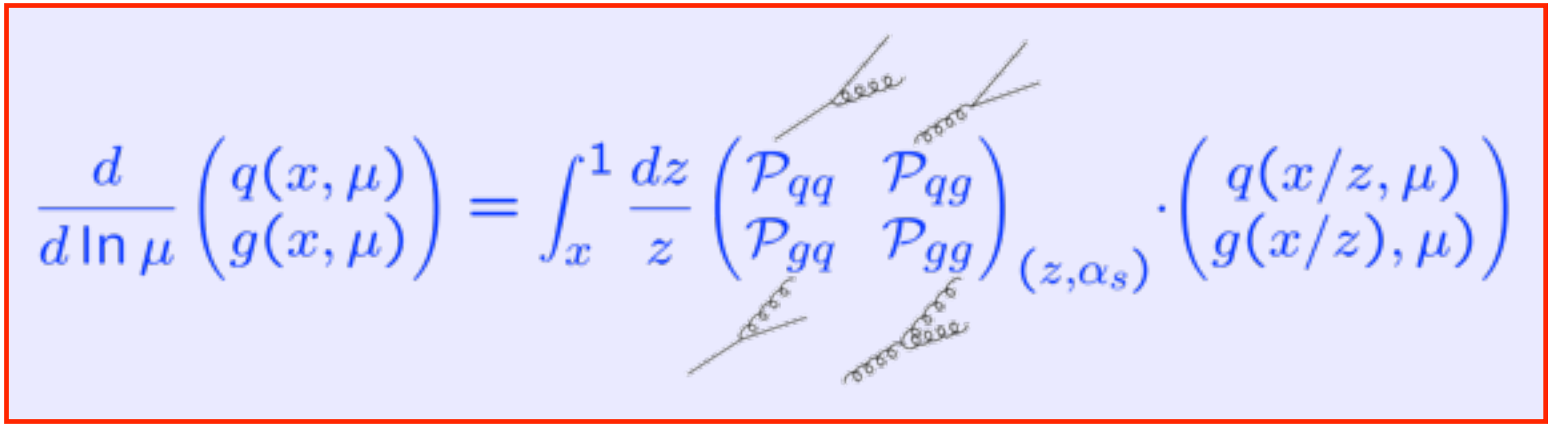
**2004**

NNLO the new emerging standard in QCD – essential for precision physics

# DGLAP evolution in full glory

**taking quarks and gluons together:** coupled integro-differential equations



$$\frac{d}{d\ln\mu}\begin{pmatrix} q(x,\mu) \\ g(x,\mu) \end{pmatrix} = \int_x^1 \frac{dz}{z} \begin{pmatrix} \mathcal{P}_{qq} & \mathcal{P}_{qg} \\ \mathcal{P}_{gq} & \mathcal{P}_{gg} \end{pmatrix}_{(z,\alpha_s)} \cdot \begin{pmatrix} q(x/z,\mu) \\ g(x/z),\mu) \end{pmatrix}$$

best solved in Mellin moment space: set of ordinary differential eqs.;
no closed solution in exp. form beyond LO (commutators of P matrices!)

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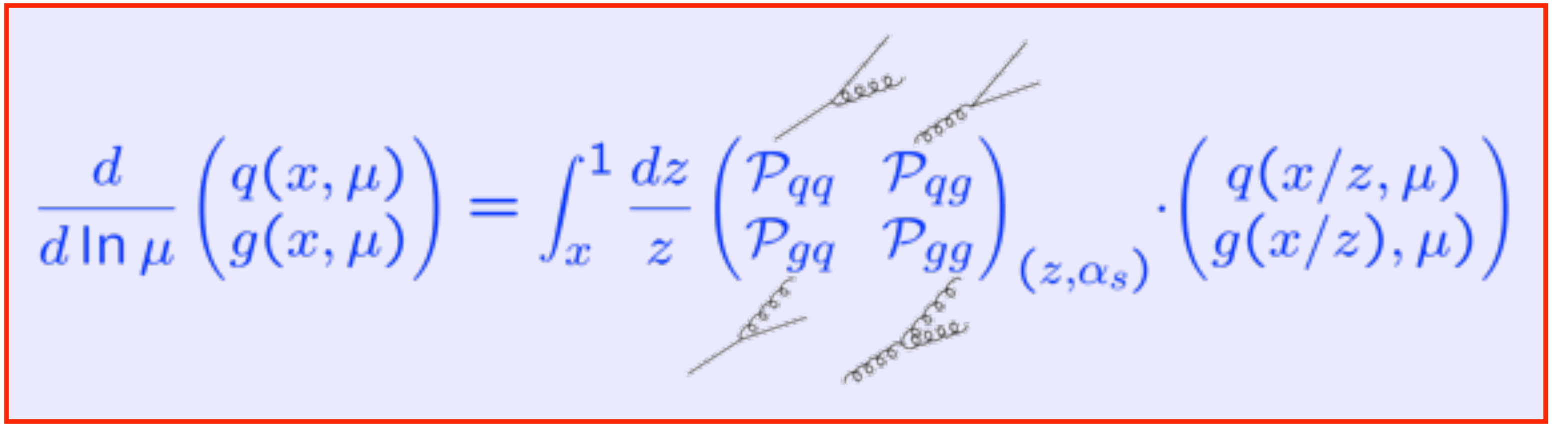

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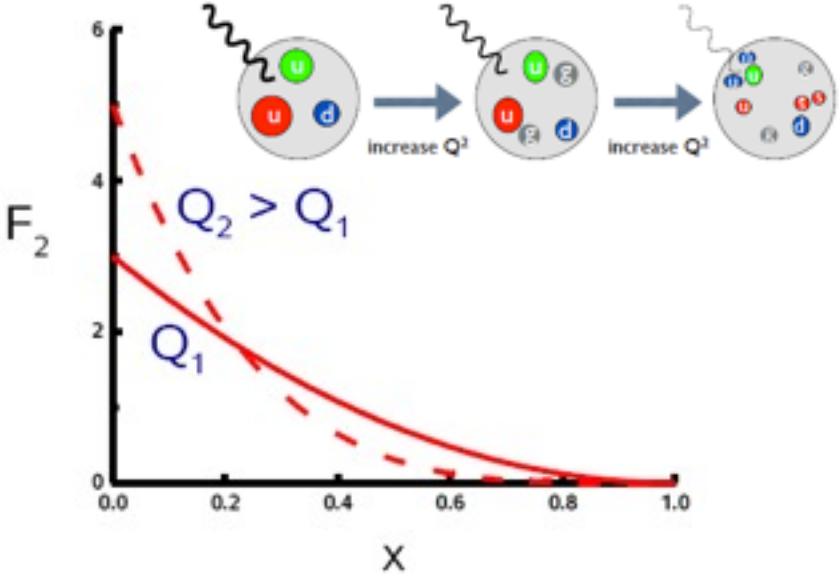
**main effect/prediction of evolution:**

partons loose energy by evolution!

- large x depletion
- small x increase



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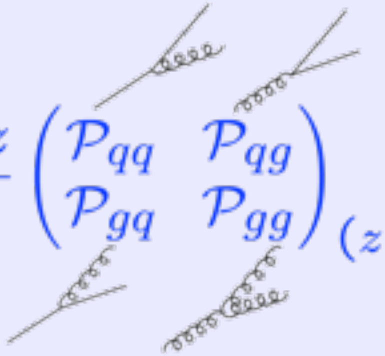

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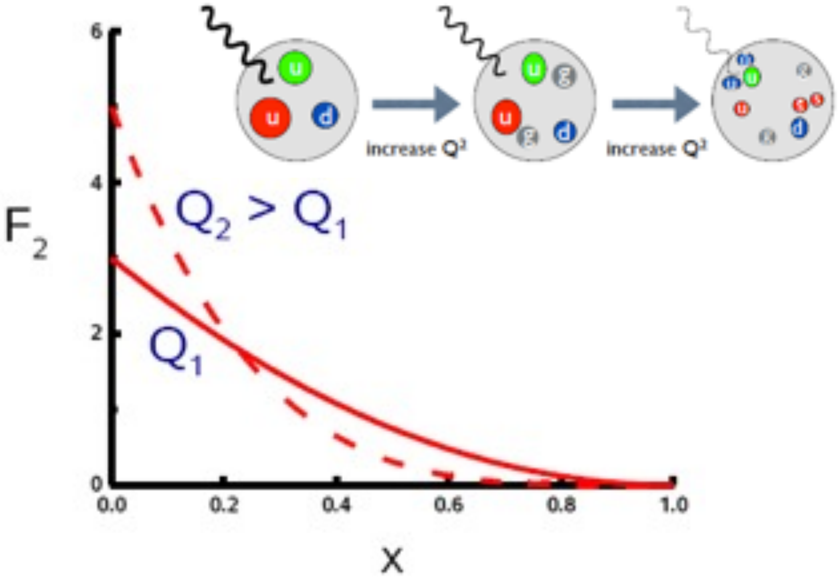
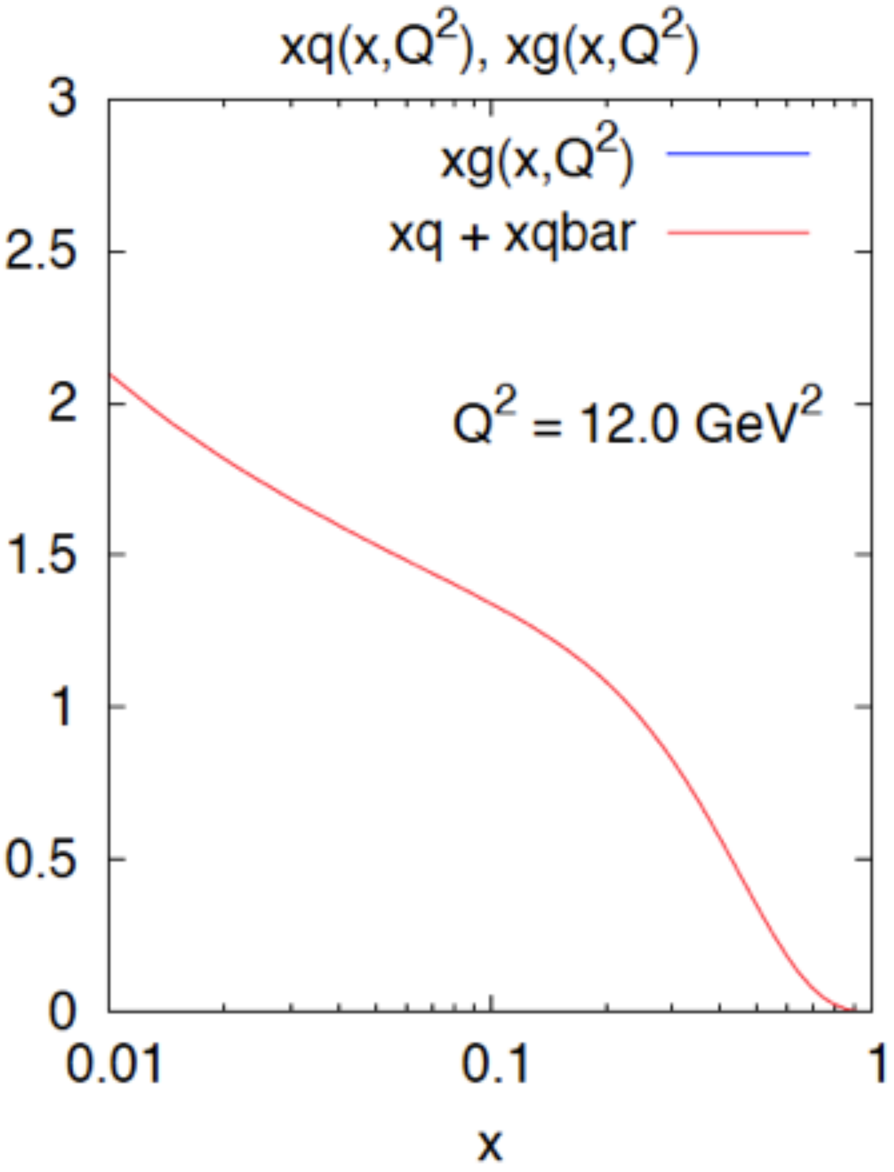
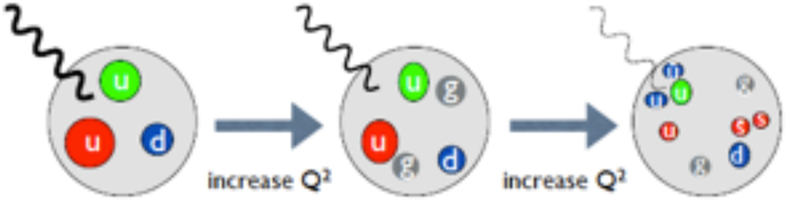
**exactly as observed in experiment**
**huge success of pQCD**



# DGLAP evolution at work: toy example





start off from just quarks, no gluons

- quarks reduced at large x
- gluons rise quickly at small x
  (which, btw, also generates sea quarks)

taken from G. Salam

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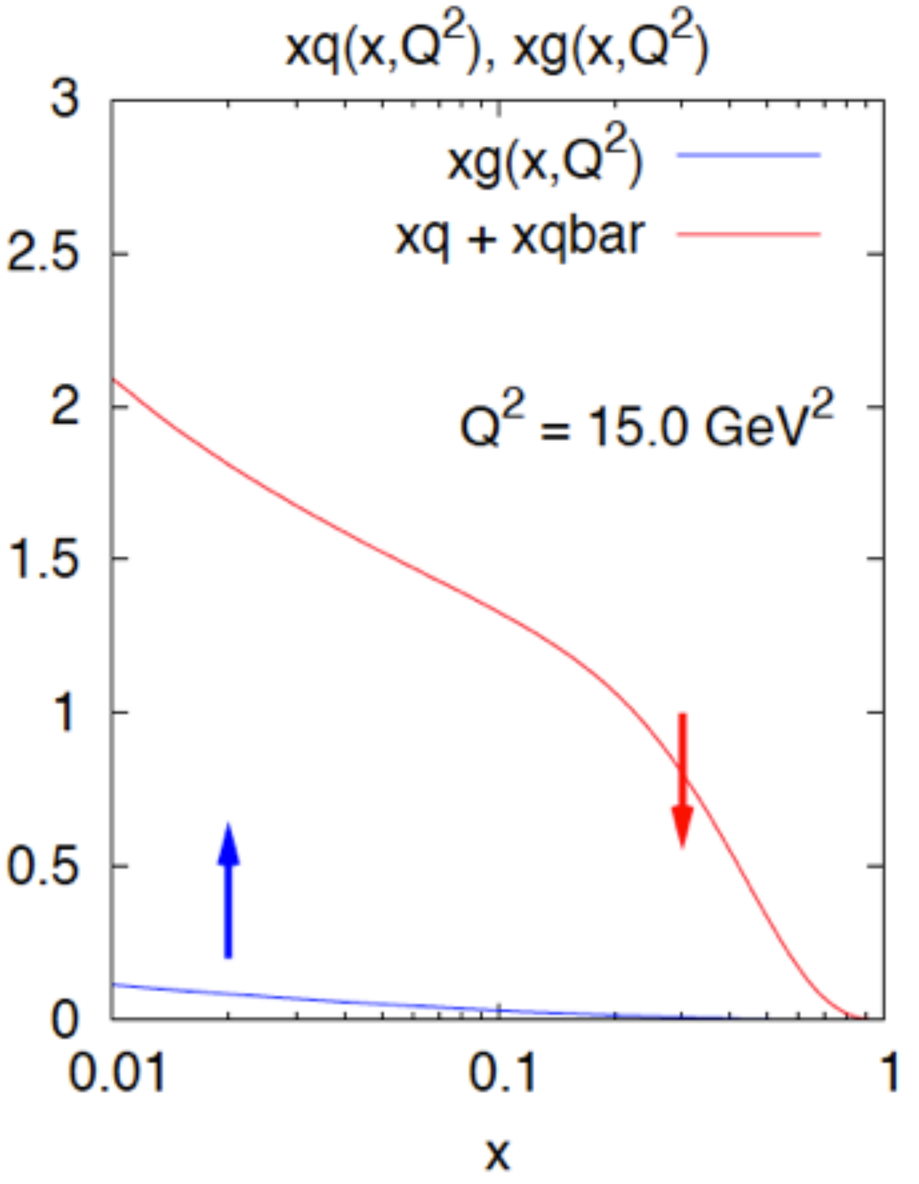

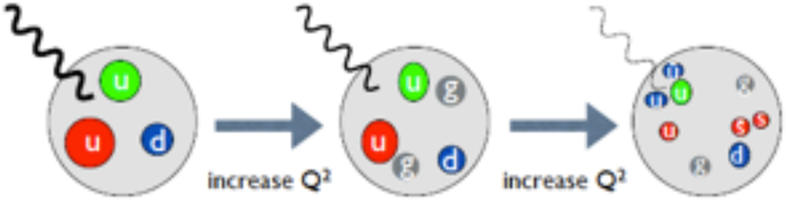

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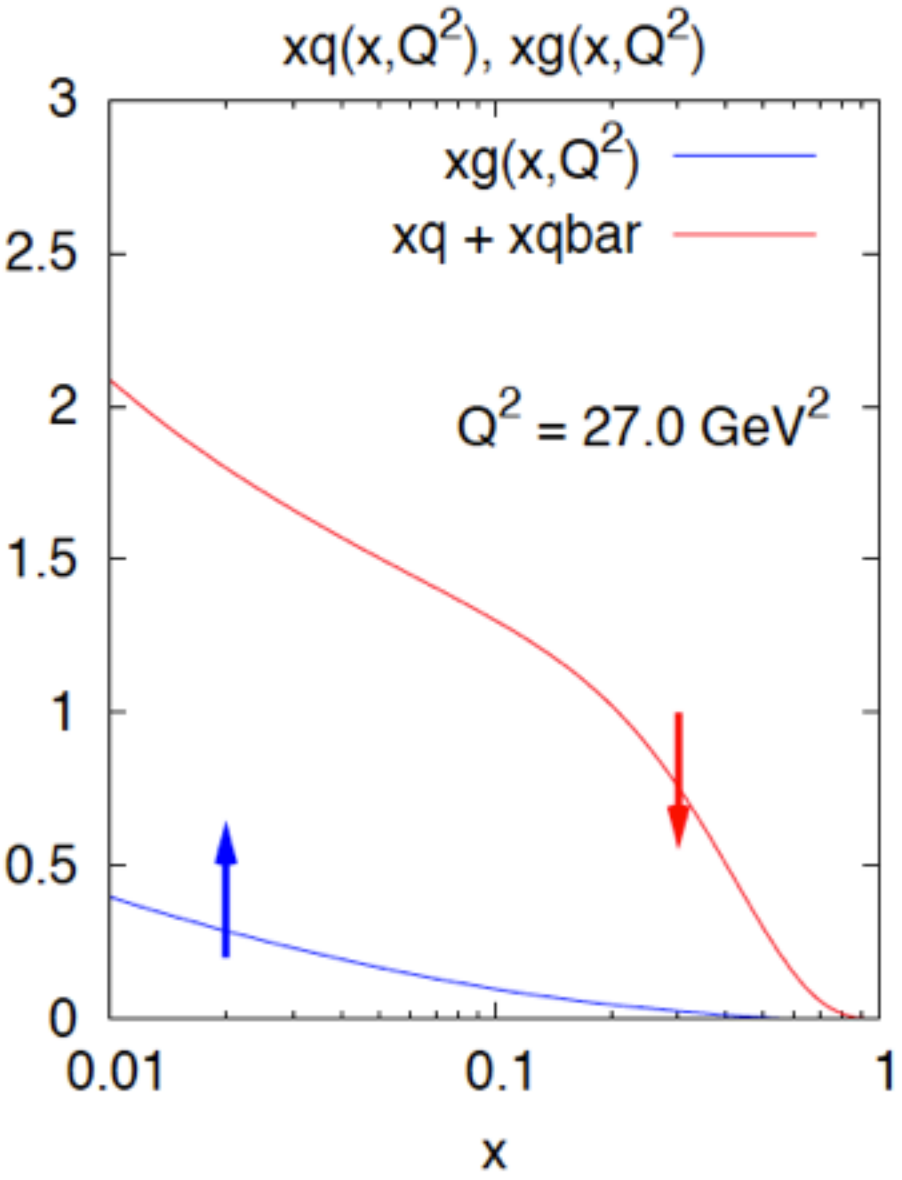

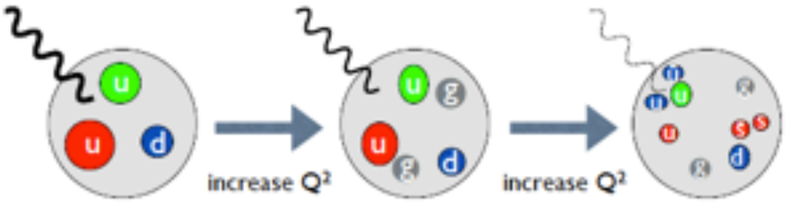

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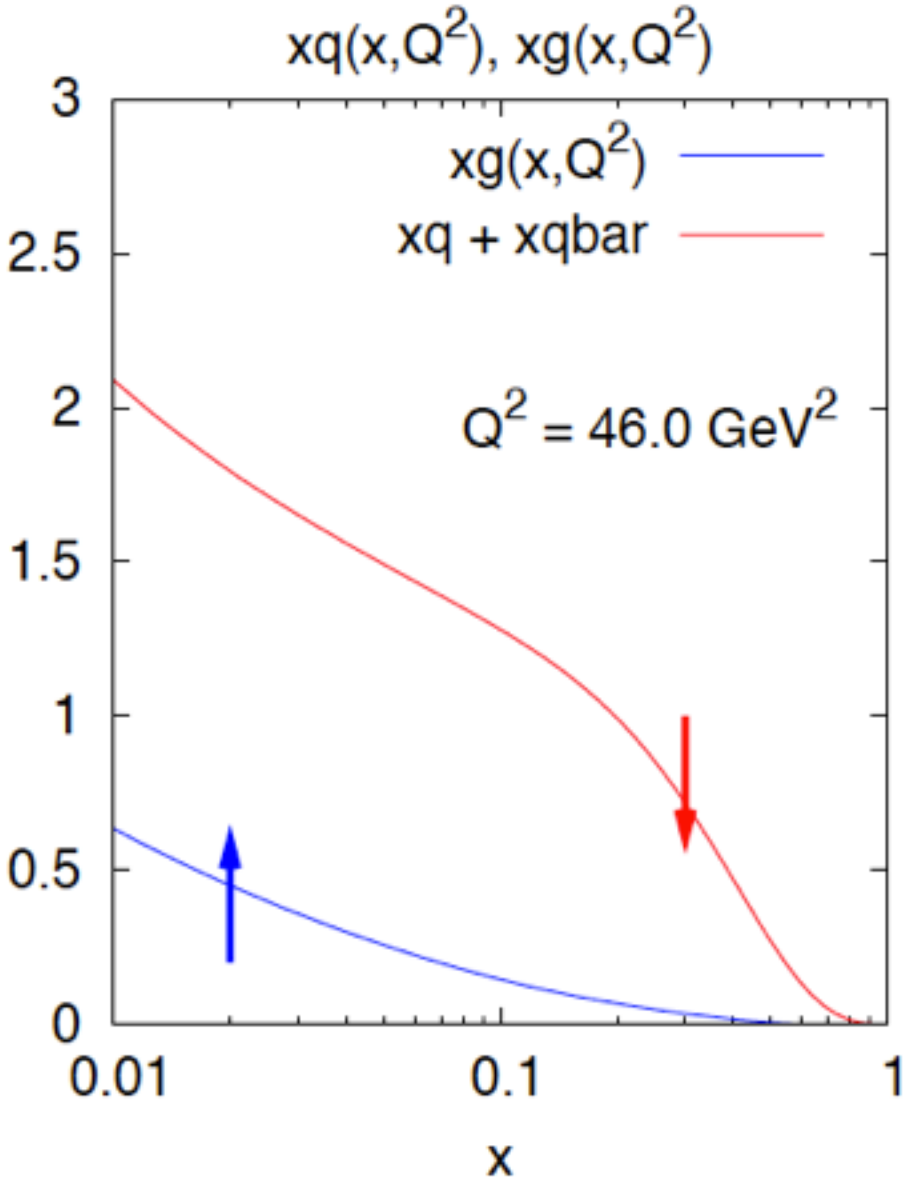

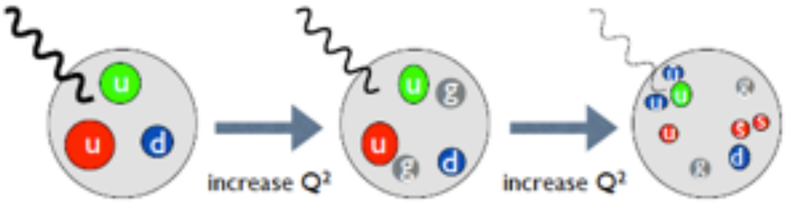

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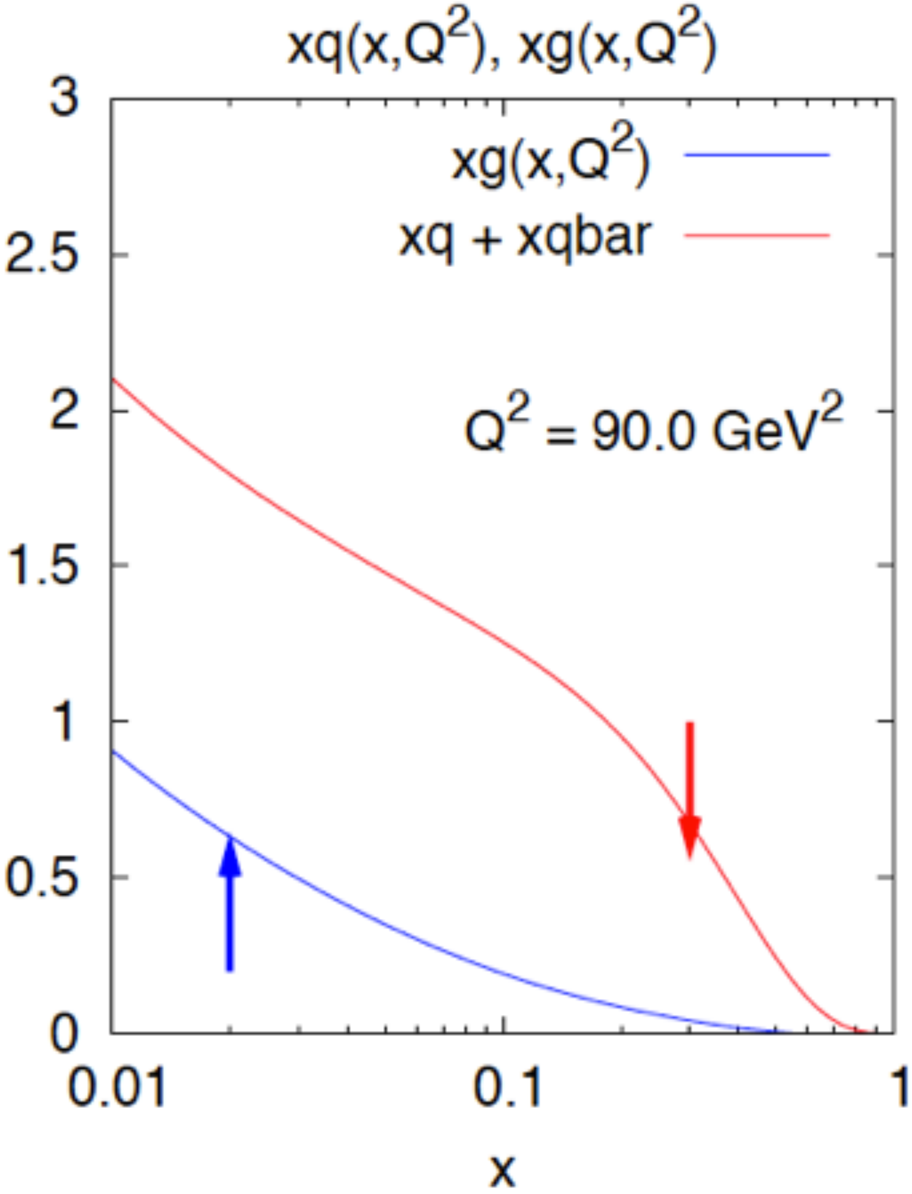

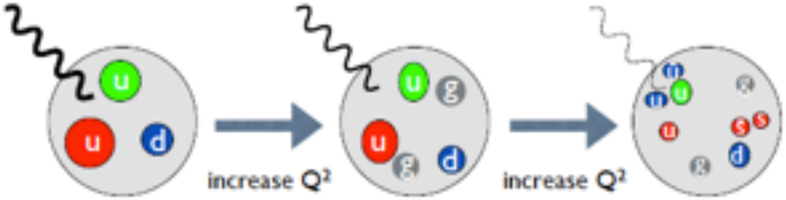

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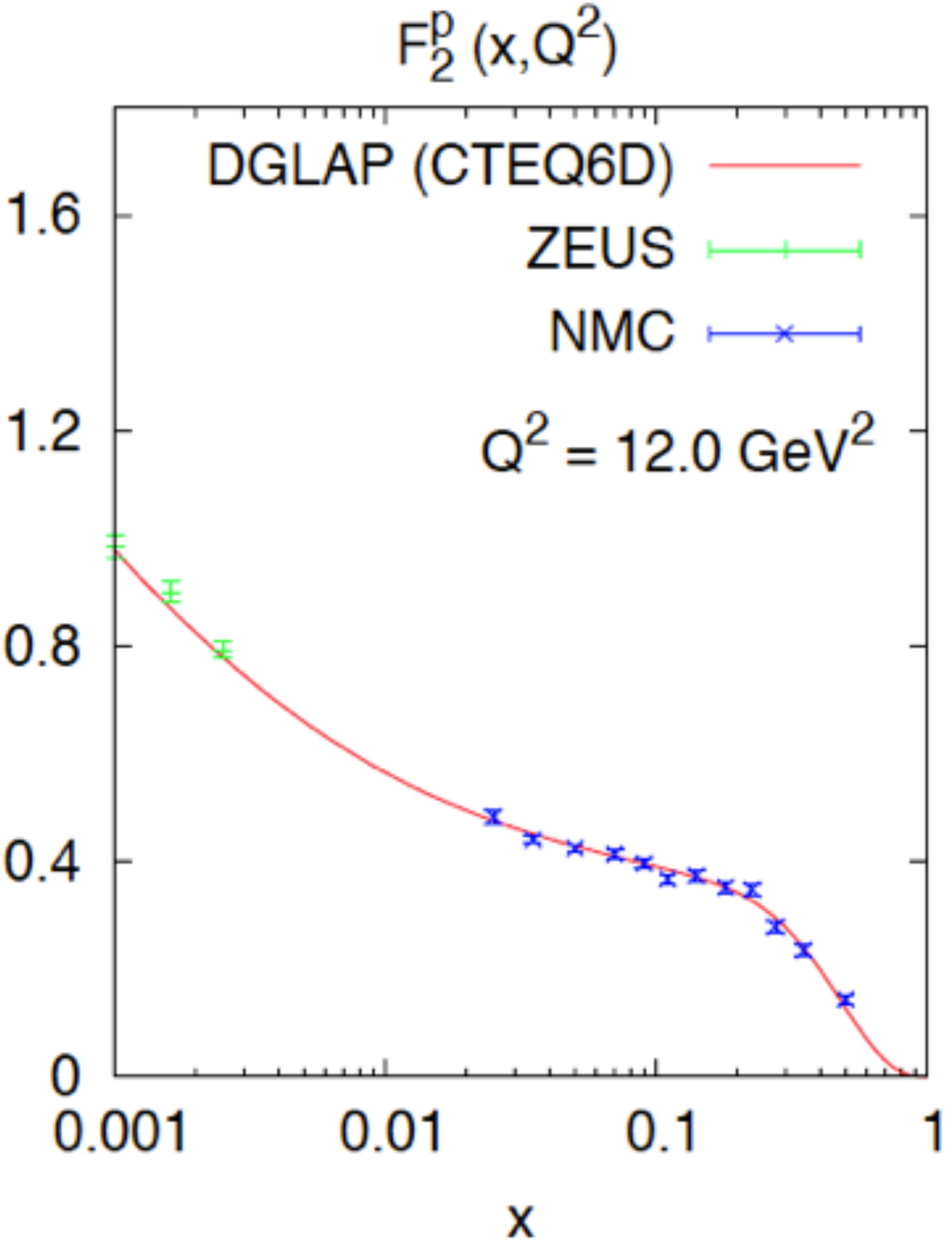

- use one of the global fits of PDFs to data by CTEQ
- steep rise of $F_2$ at small x (due to gluon evolution)

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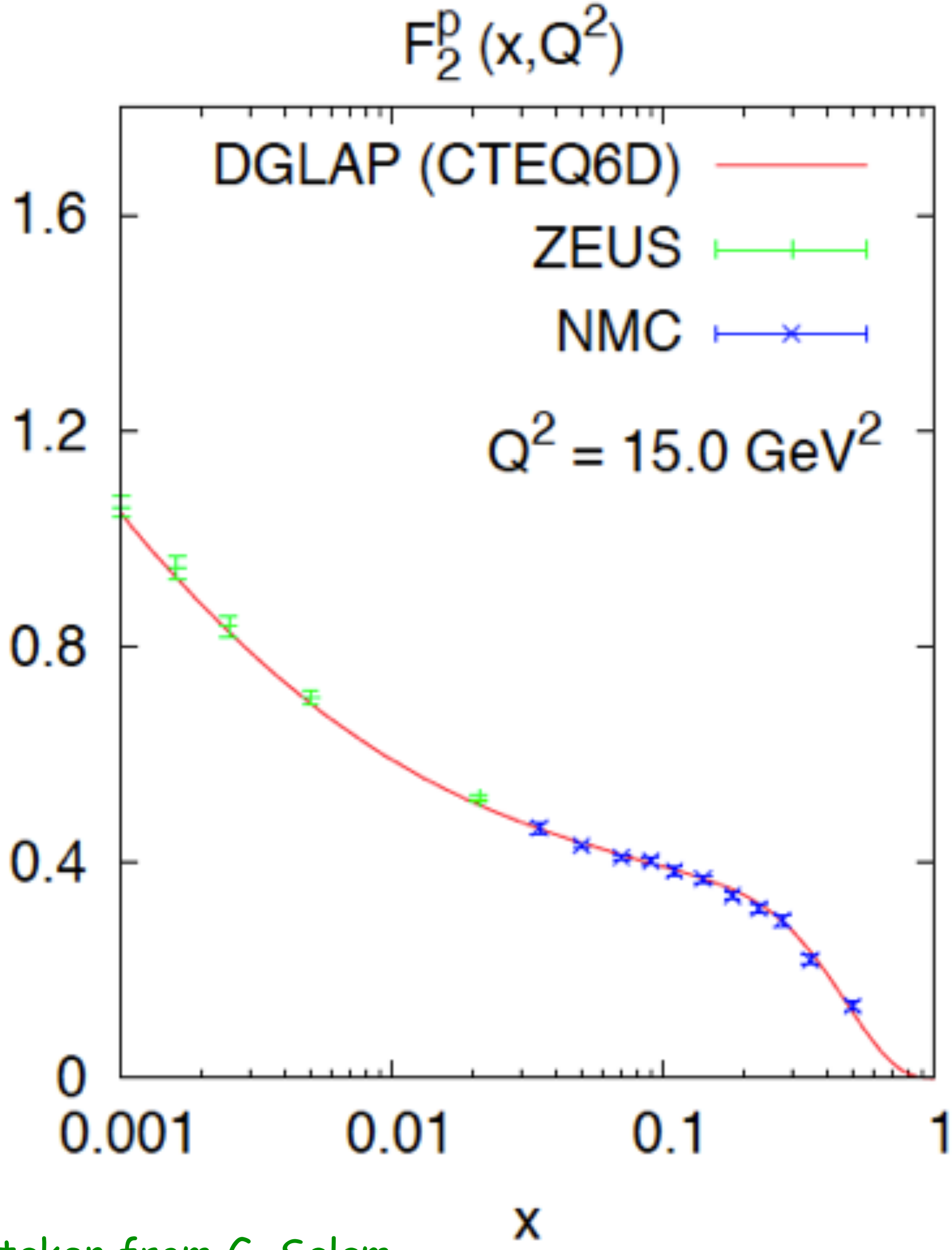

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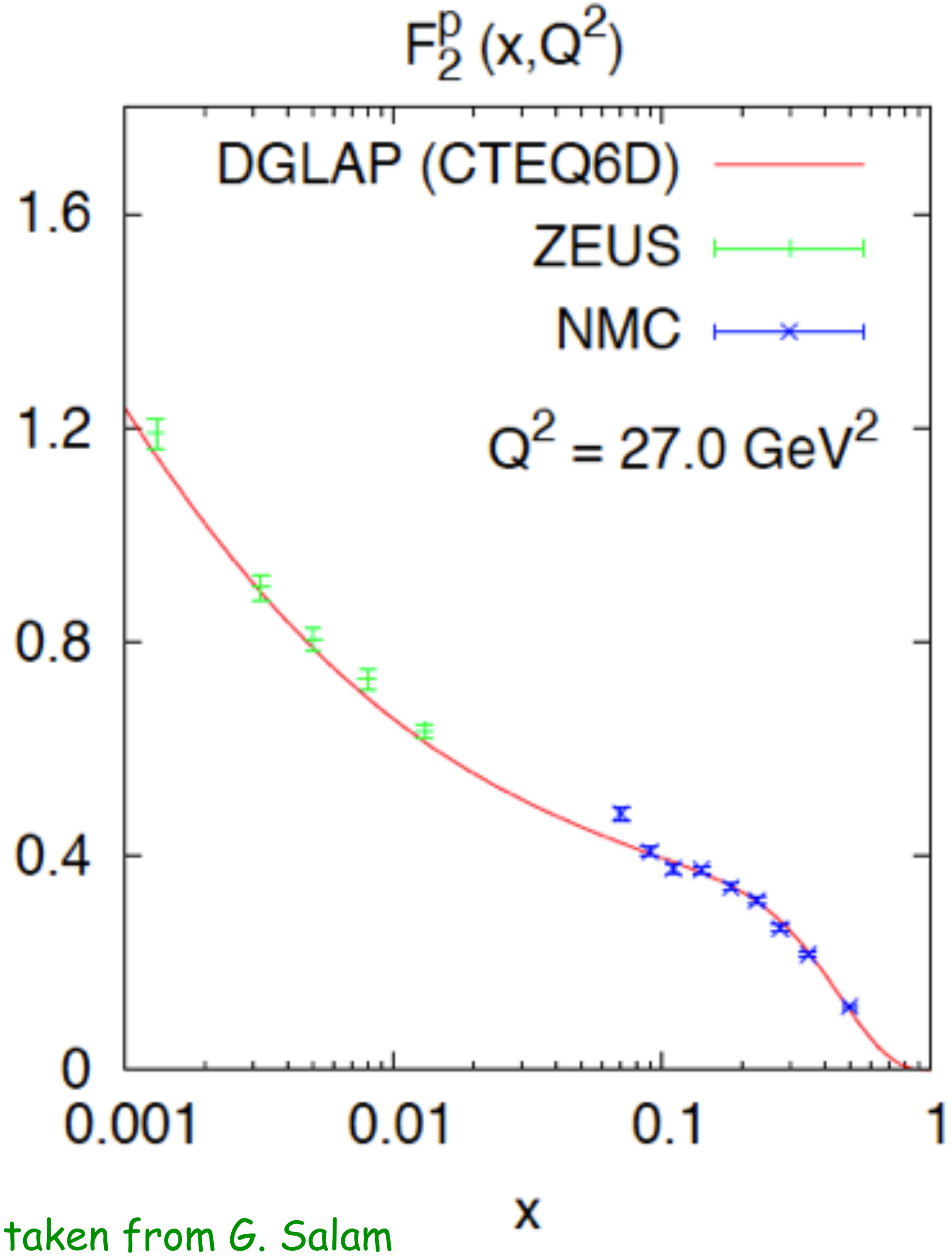

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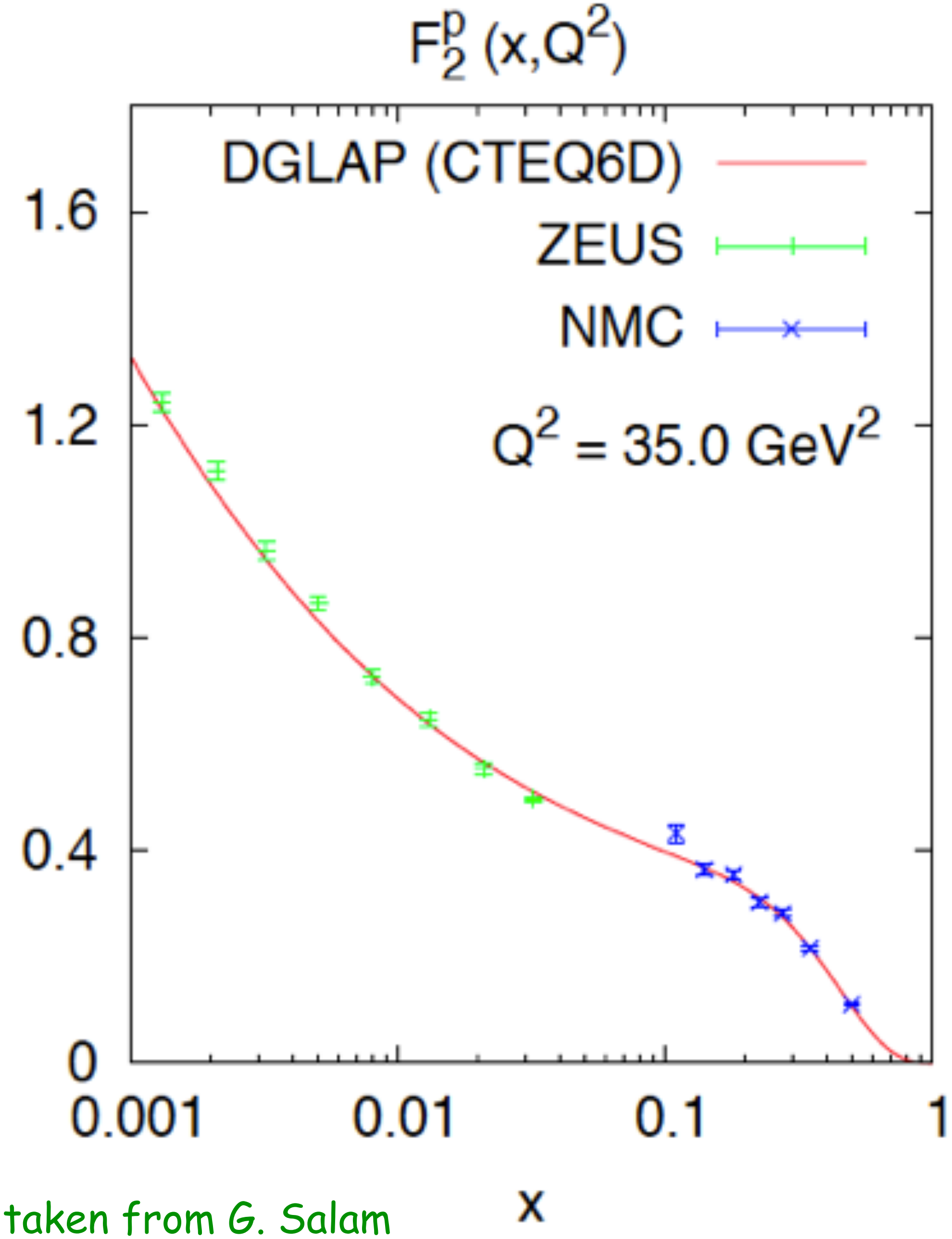

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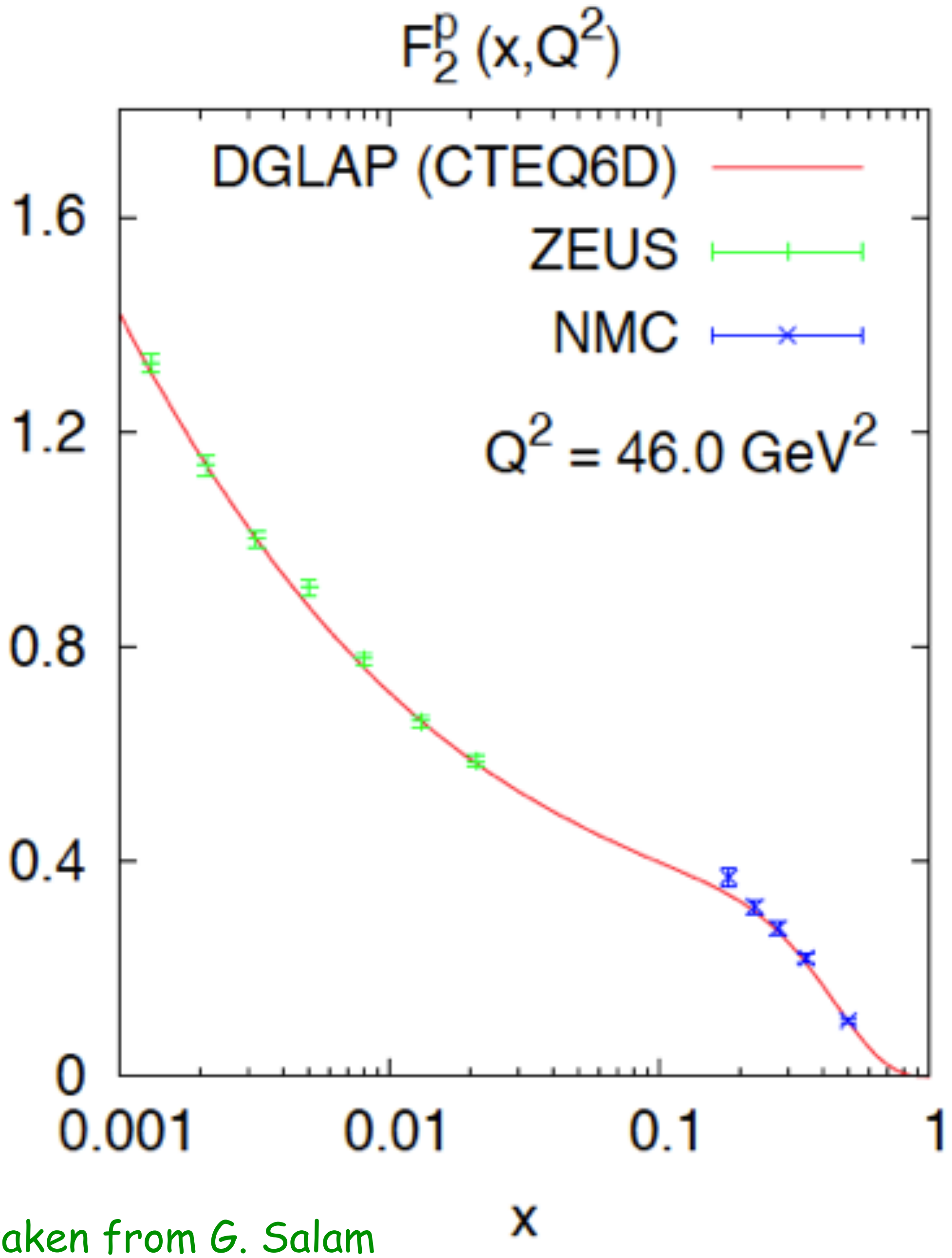

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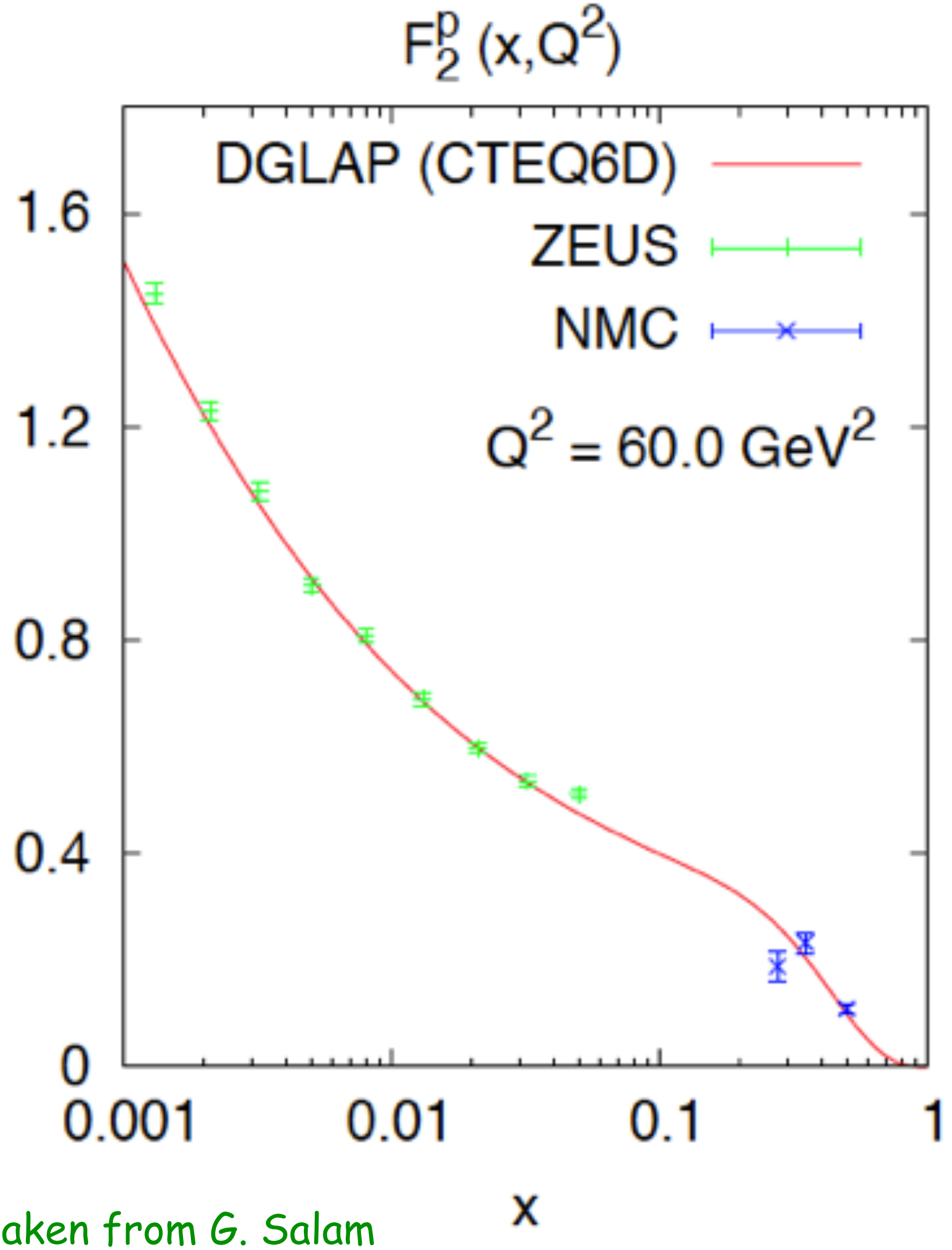

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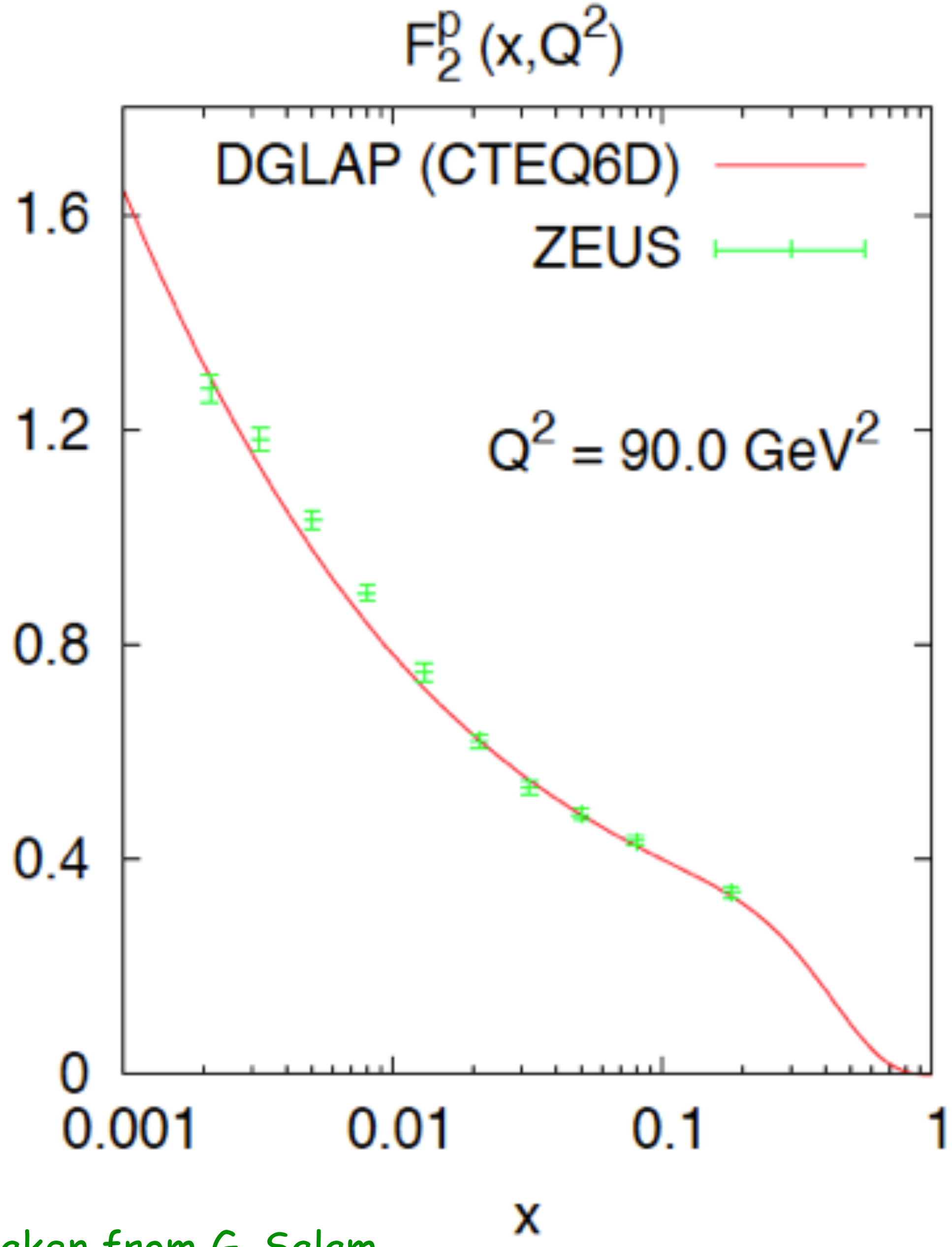

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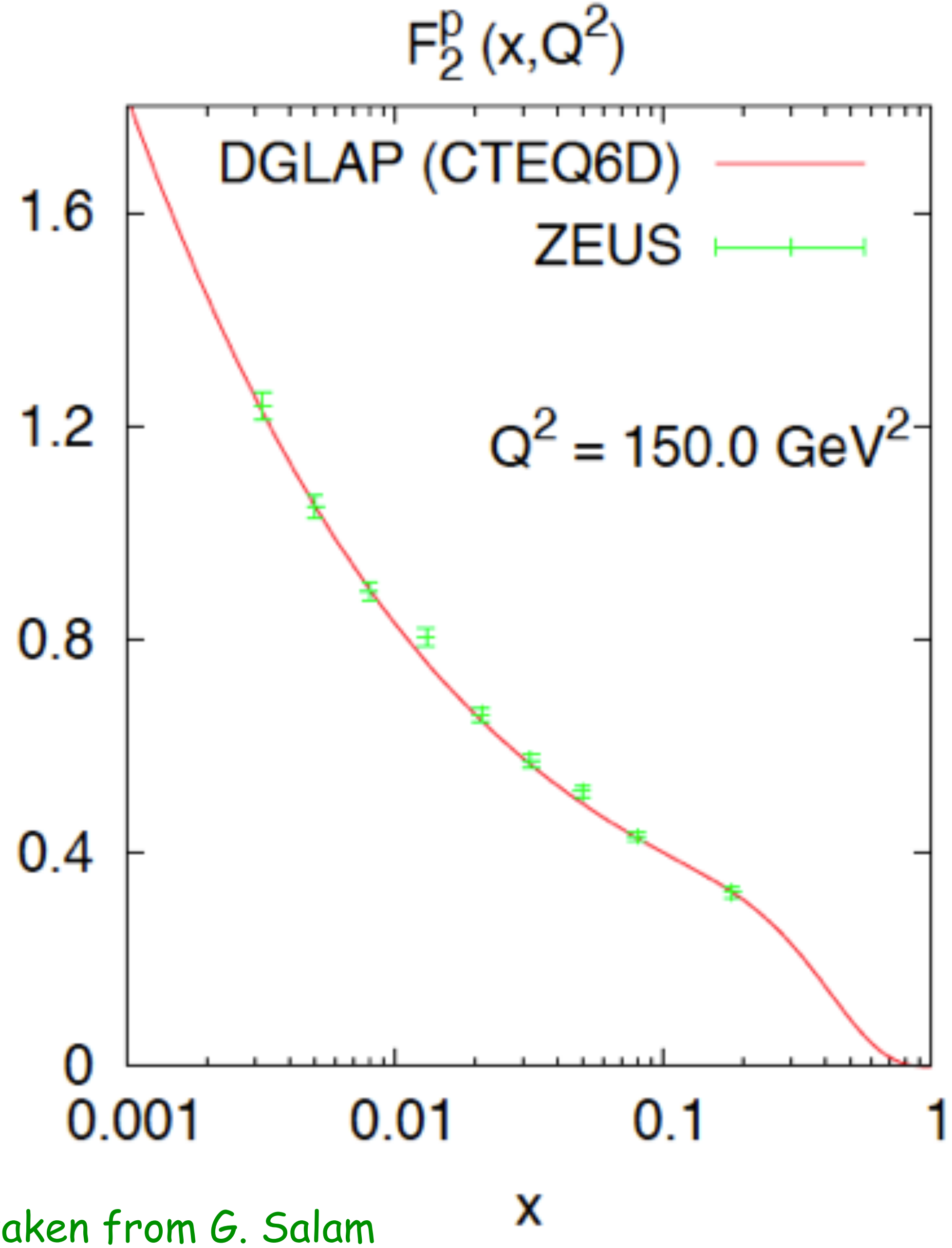

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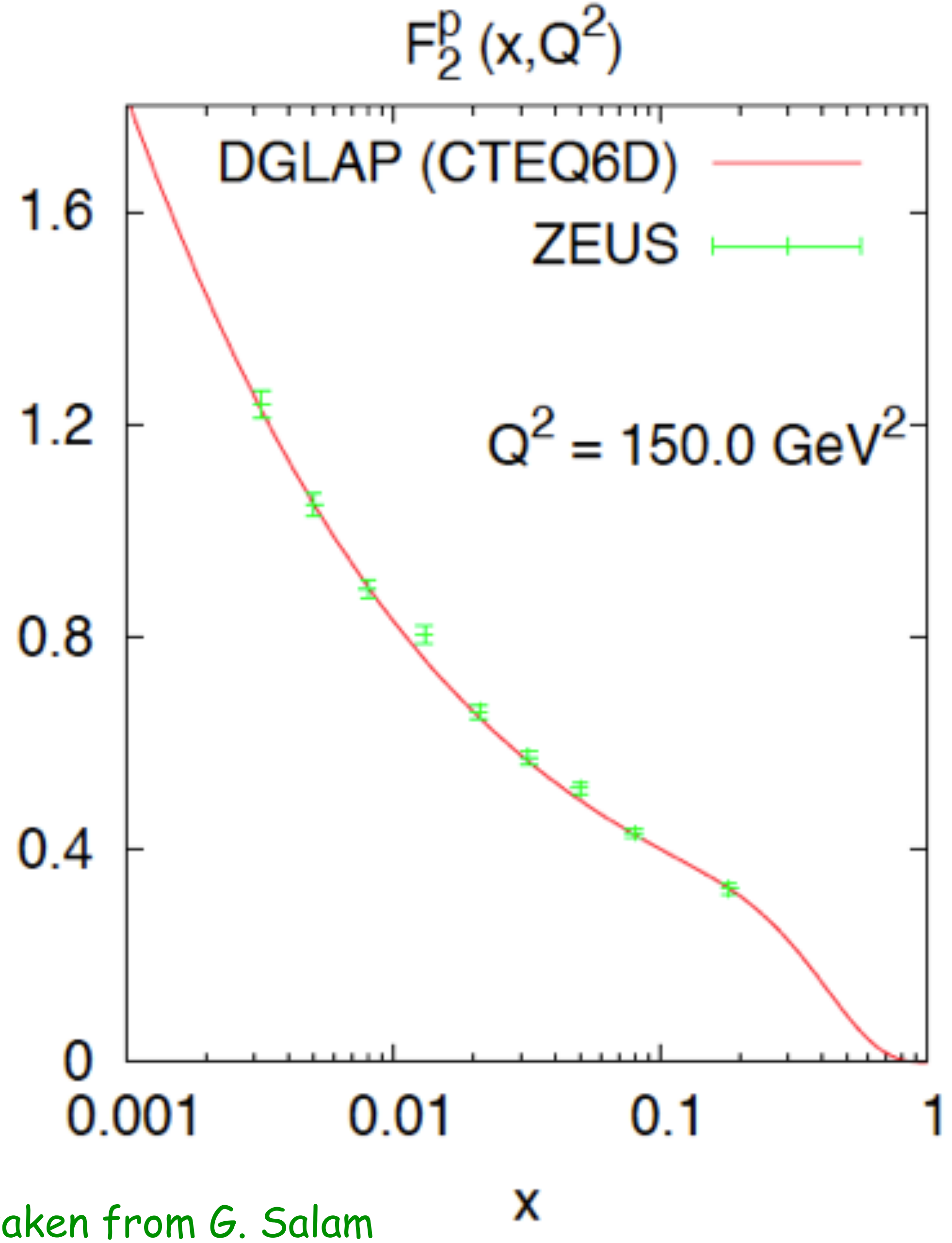

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**major success of pQCD and DGLAP evolution**

taken from G. Salam

# factorization in hadron-hadron collisions

**What happens when two hadrons collide ?**

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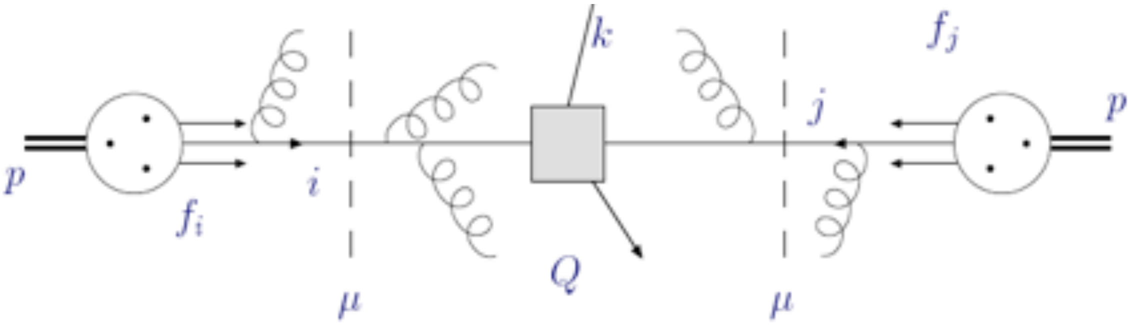
straightforward generalization of the concepts discussed so far:


**jets, hadrons, heavy quarks, ...**

# factorization in hadron-hadron collisions

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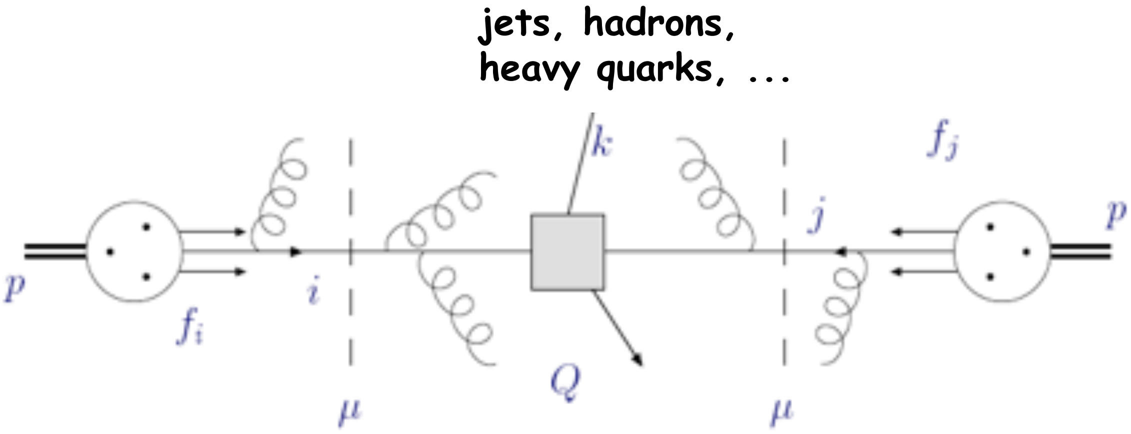


$$d\sigma = \sum_{ij} \int dx_i dx_j \, f_i(x_i, \mu^2) \, f_j(x_j, \mu^2) \, d\hat{\sigma}_{ij}(\alpha_s(\mu_r), Q^2, \mu^2, x_i, x_j)$$

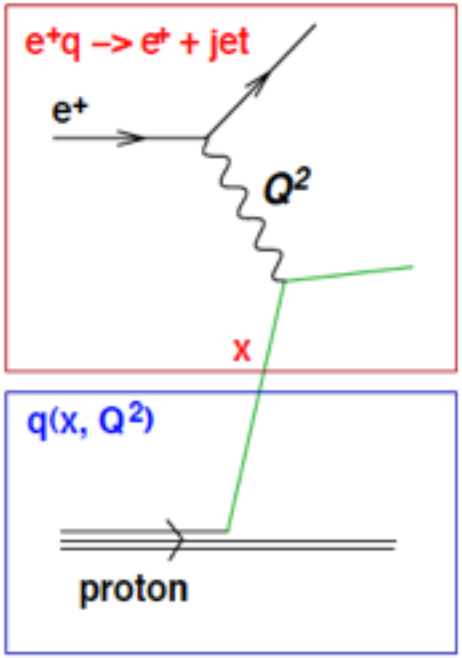
**non-perturbative but universal PDFs** $\xleftrightarrow[\text{by } \mu]{\text{linked}}$ **hard scattering of two partons → pQCD**

# factorization at work

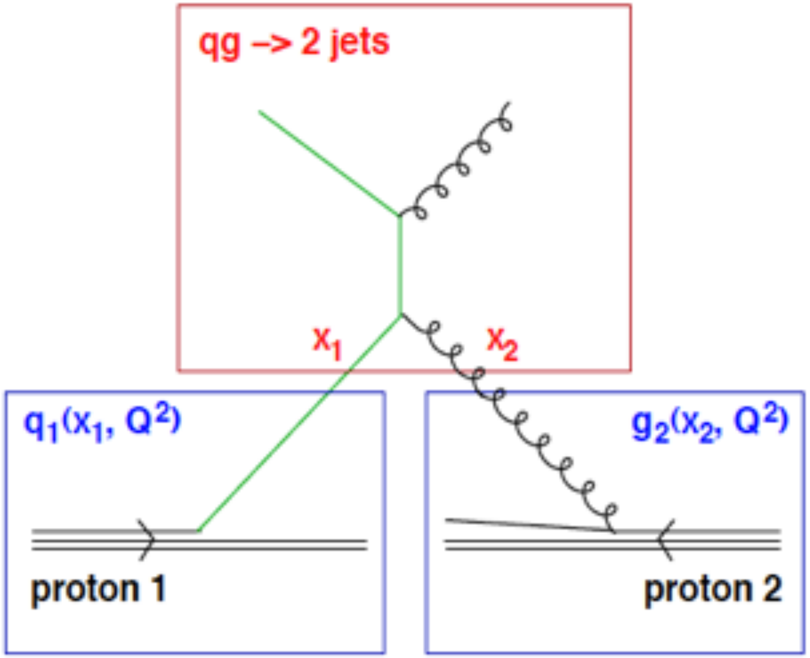
key assumption that a cross section factorizes into

- hard (perturbatively calculable) process-dep. partonic subprocesses
- non-perturbative but universal parton distribution functions

has great **predictive power** and can be challenged experimentally:



$\sigma_{ep} = \sigma_{eq} \otimes q$



$\sigma_{pp\to 2\,jets} = \sigma_{qg\to 2\,jets} \otimes q_1 \otimes g_2 + \cdots$

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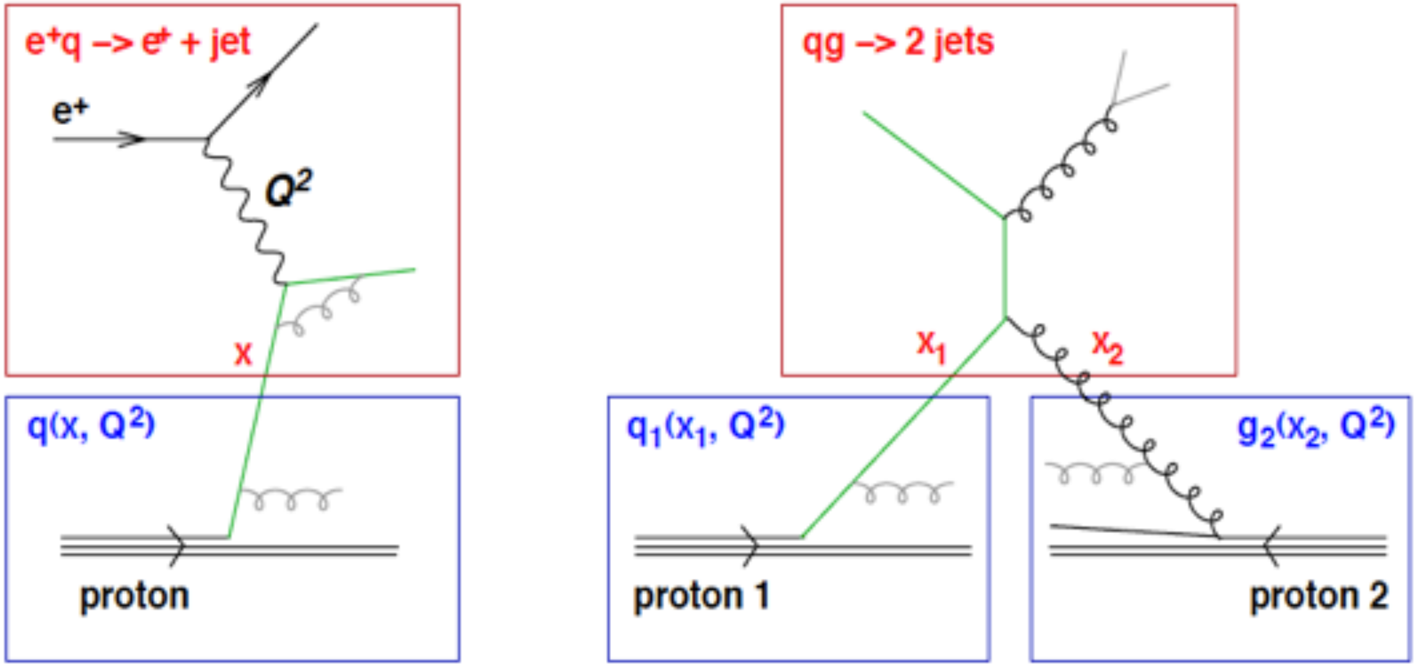

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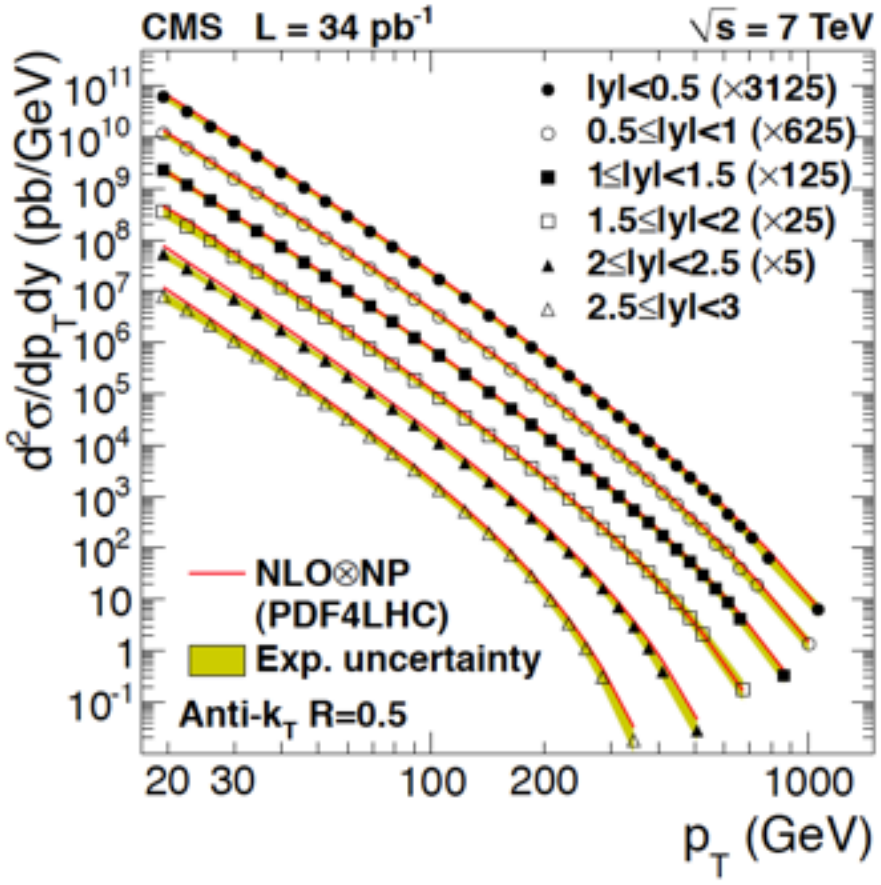
# factorization: so far a success story



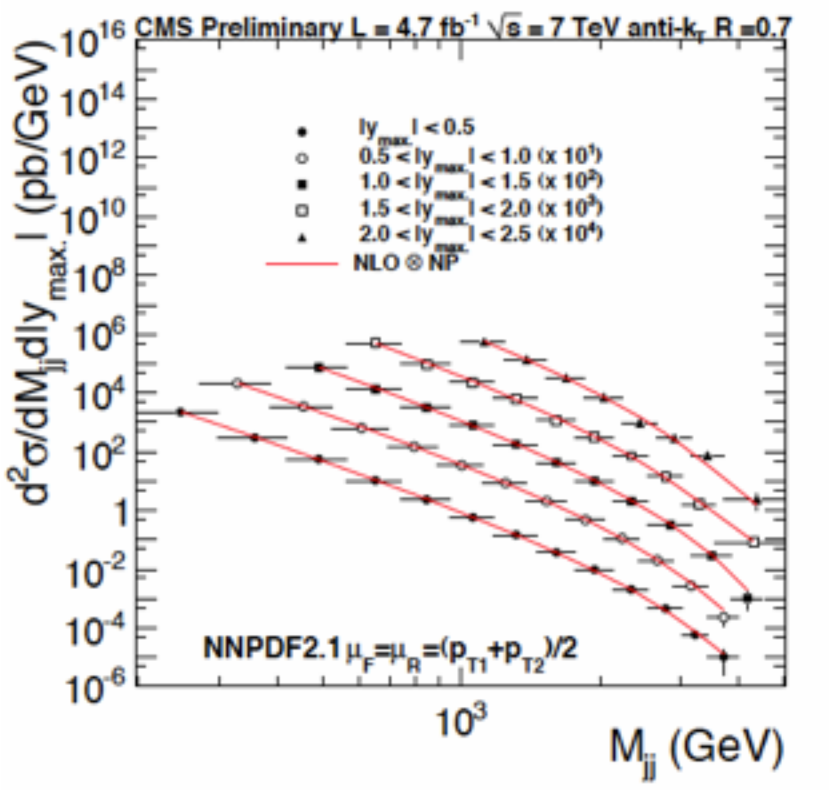
**two recent examples from the LHC:**

1-jet and di-jet cross sections

many other final-states available

$$y = \ln\tan\frac{\theta}{2} \sim \frac{1}{2}\ln\frac{x_1}{x_2} \qquad M = \sqrt{x_1 x_2 s}$$

$$x_1 = \frac{M}{\sqrt{s}}e^{+y} \qquad x_2 = \frac{M}{\sqrt{s}}e^{-y}$$



results now start to being used in global fits to constrain PDFs

**particularly sensitive to gluons**

$$gg \rightarrow gg \qquad gq \rightarrow gq$$

# proofs of factorization

- to prove the validity of **factorization to all orders** of pQCD is a highly theoretical and technical matter

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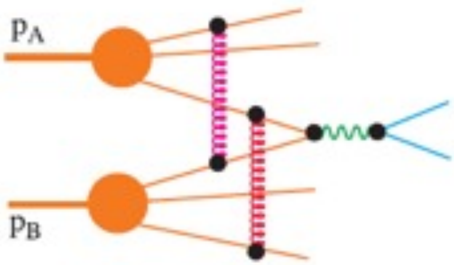
- **serious proofs** exist only for a limited number of processes such as DIS and Drell-Yan **Libby, Sterman; Ellis et al.; Amati et al.; Collins et al.;...**



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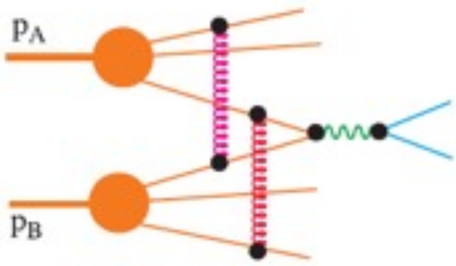

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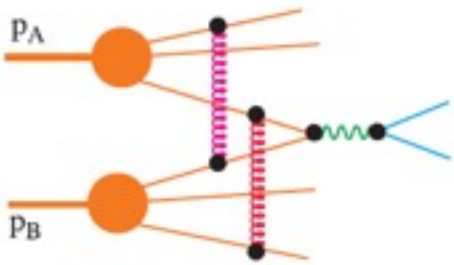

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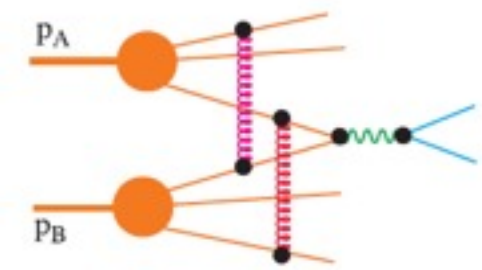

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---

**recall:** the renormalizibility of a non-abelian gauge theory like QCD was demonstrated by 't Hooft and Veltman

1999

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**recap:** salient features of pQCD

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## recap: salient features of pQCD

- strong interactions, yet perturbative methods are applicable
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keys to resolve the apparent dilemma:

- asymptotic freedom
- infrared safety
- factorization theorems & renormalizibility

# pQCD: a tool for the most violent collisions

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# pQCD: a tool for the most violent collisions

**“soft stuff”**: difficult!

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**“underlying event”**: more than difficult

# to take home from this part of the lectures

- factorization = isolating and absorbing long-distance singularities accompanying identified hadrons into parton densities (initial state) and fragmentation fcts. (final state)
- factorization and renormalization introduce arbitrary scales
  $\rightarrow$ powerful concept of renormalization group equations
  $\rightarrow$ $\alpha_s$, PDFs, frag. fcts. depend on energy/resolution
- PDFs (and frag. fcts) have definitions as bilocal operators
- hard hadron-hadron interactions factorize as well: $f \otimes f \otimes d\sigma$
- strict proofs of factorization only for limited class of processes

# *unofficial* Part IV

## some applications & advanced topics

scales and theoretical uncertainties; Drell-Yan process
small-x physics; global QCD analysis; resummations

# 1

# the Whys and Hows of NLO Calculations & Beyond

# why go beyond LO (and even NLO)?

**recall factorization theorem for hadronic processes:**

$$d\sigma = \sum_{ij} \int dx_i dx_j \, f_i(x_i, \mu^2) \, f_j(x_j, \mu^2) \, d\hat{\sigma}_{ij}(\alpha_s(\mu_r), Q^2, \mu^2, x_i, x_j)$$

non-perturbative but universal PDFs $\xleftrightarrow[\text{by } \mu]{\text{linked}}$ hard scattering of two partons → pQCD

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**caveat:** we work with a perturbative series truncated at LO, NLO, NNLO, …

→ at any fixed order N there will be a **residual scale dependence** in our theoretical prediction

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**simplest example:**

$e^+e^- \to$ hadrons

$$\frac{d}{d \ln \mu_r} \sum_{n=1}^{N} c_n(\mu_r) \alpha_s^n(\mu_r) \sim \mathcal{O}\left(\alpha_s^{N+1}(\mu_r)\right)$$

applies in general also for $\mu_f$

**uncertainty is formally of higher order**

-> gets smaller if higher orders are known

# explicit example: scale dependence of $e^+e^-$ --> jets

**recall:** at NLO we have $\sigma^{\mathrm{NLO}}(\mu_R) = \sigma_{q\bar{q}}\left(1 + c_1\,\alpha_s(\mu_R)\right)$

LO result

NLO coefficient independent of scale

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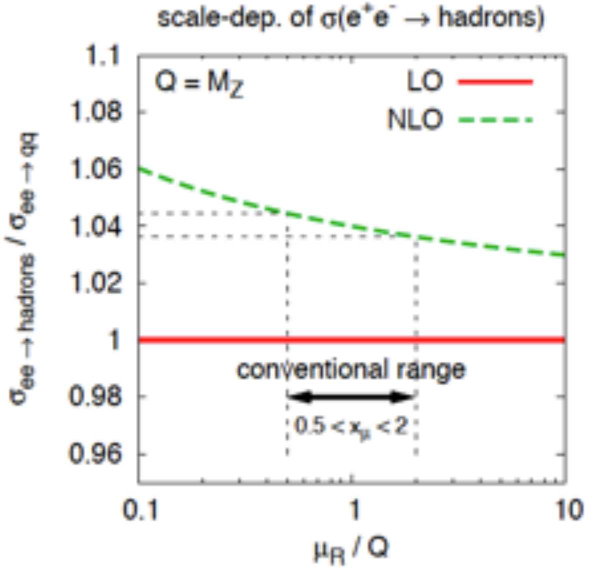

LO is a pure el-mag process, no $\alpha_s$ , no scales

# explicit example - cont'd

next calculate full NNLO result:

$$\sigma^{\mathrm{NNLO}}(\mu_R) = \sigma_{q\bar{q}}\,\left[1 + c_1\,\alpha_s(\mu_R) + c_2(\mu_R)\alpha_s^2(\mu_R)\right]$$

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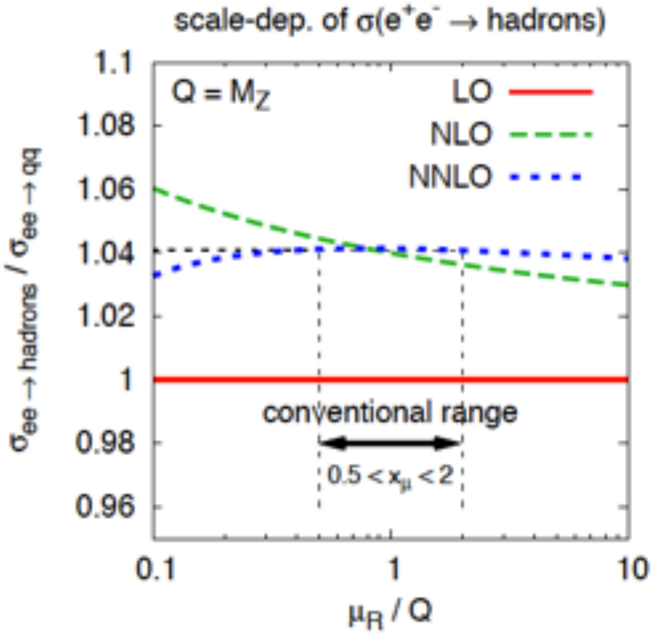

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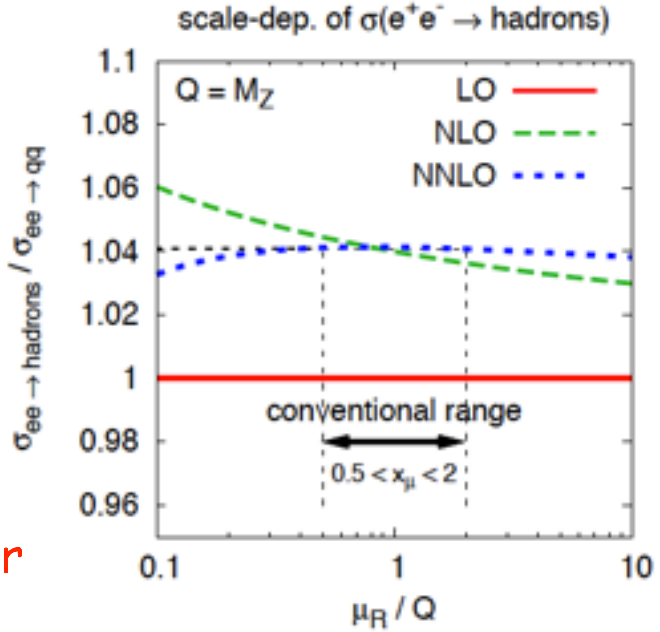

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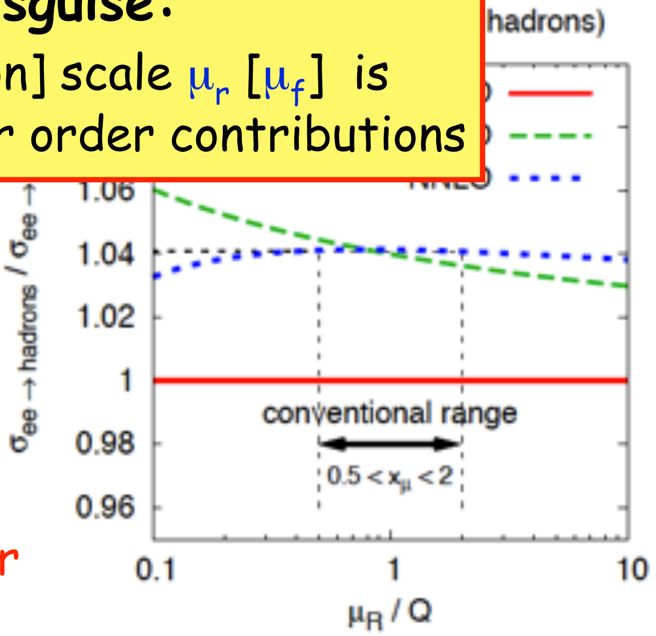
**scale "ambiguity" is a blessing in disguise:**
varying the renormalization [factorization] scale $\mu_r$ [$\mu_f$] is a way of guessing the uncalculated higher order contributions

such that the residual scale dependence is now $O(\alpha_s^3)$



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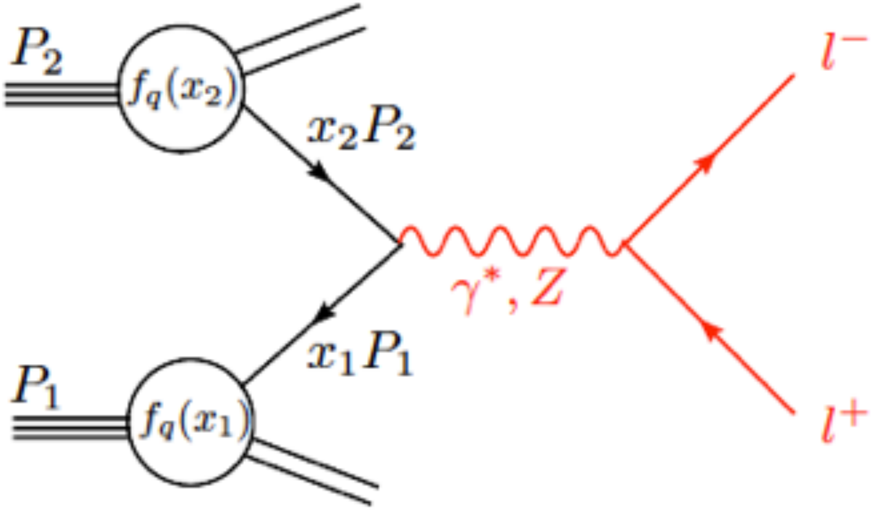
# example from hadronic collisions

**take the "classic" Drell Yan process**



- dominated by quarks in the initial-state
- at LO no colored particles in the final-state
- clean experimental signature
- at LO an electromagnetic process (low rate)
- one of the best studied processes (known to NNLO)

**as "clean" as it can get at a hadron collider**

# uncertainties for the Drell Yan process – cont'd

**at NLO:**

LO piece

$$\sigma^{\mathrm{NLO}}_{pp\to Z} = \sum_{i,j} \int dx_1 dx_2\, f_i(x_1,\mu_F^2)\, f_j(x_2,\mu_F^2)\, [\hat{\sigma}_{0,ij\to Z}(x_1,x_2) + \\ + \alpha_s(\mu_R)\hat{\sigma}_{1,ij\to Z}(x_1,x_2,\mu_F)]$$

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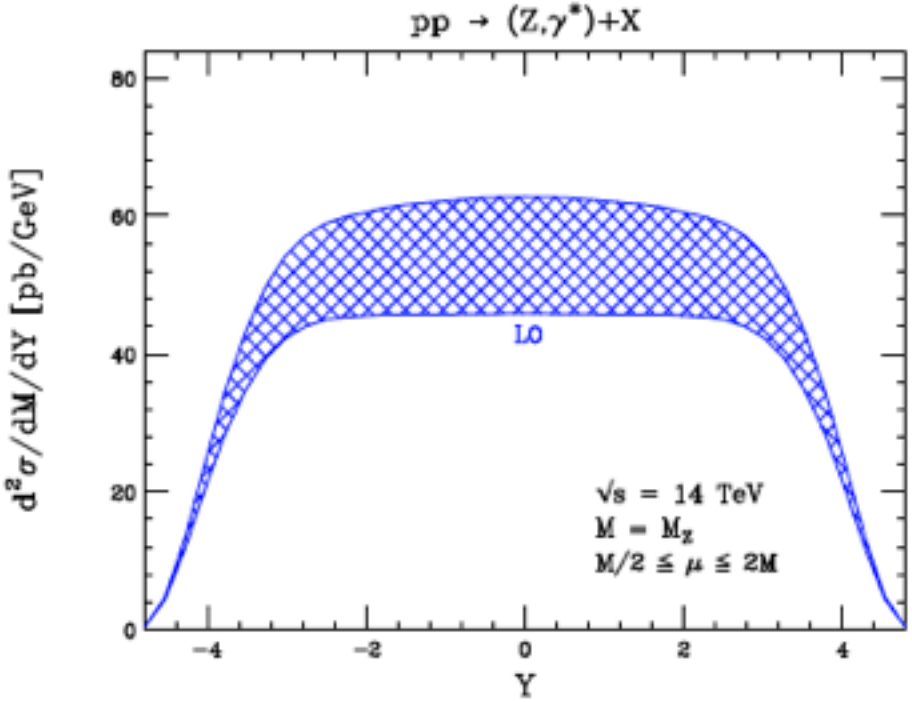

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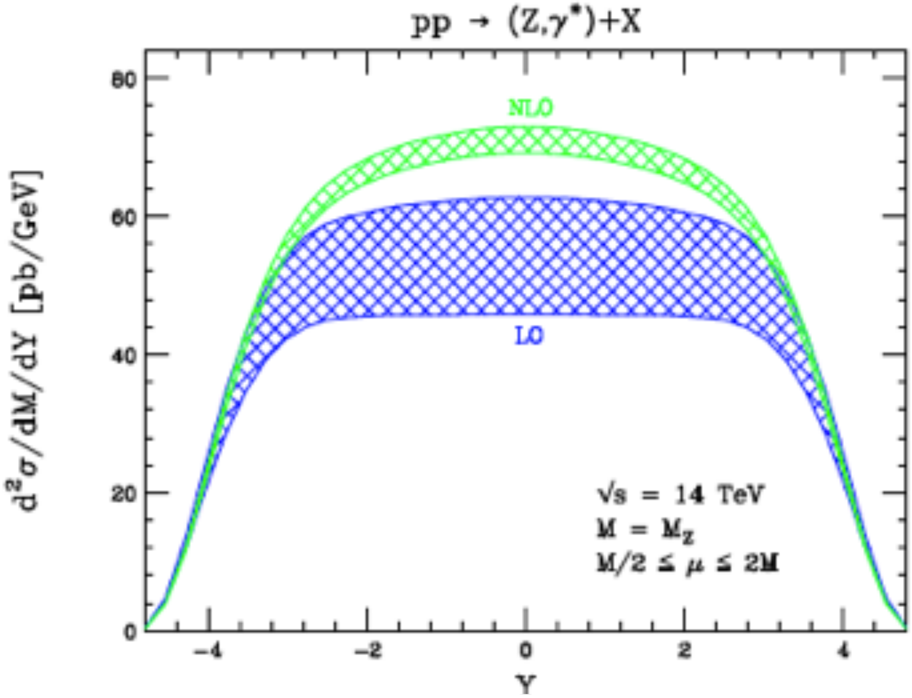

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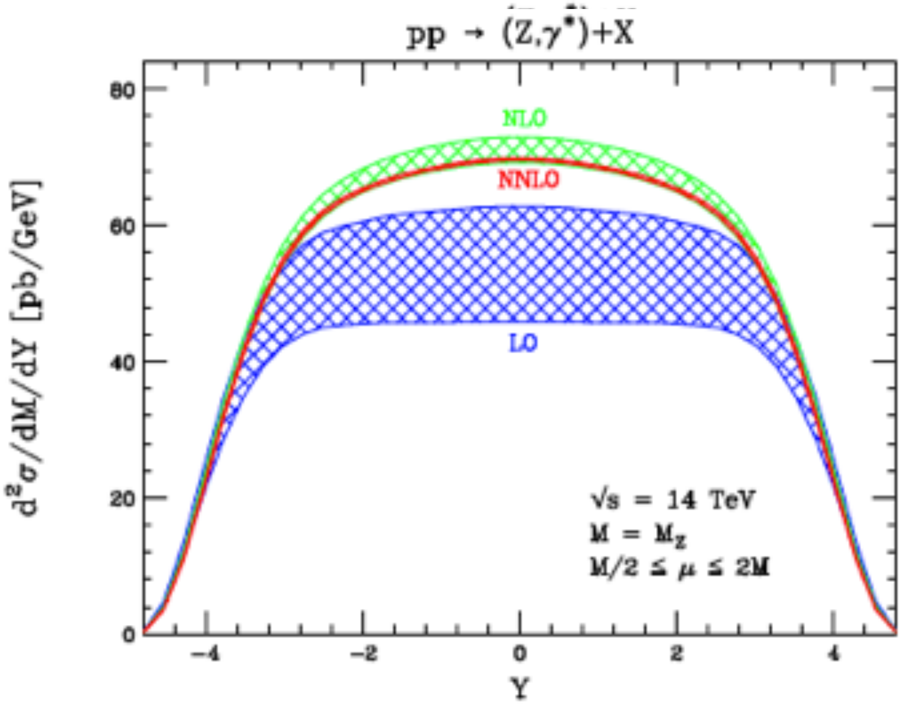

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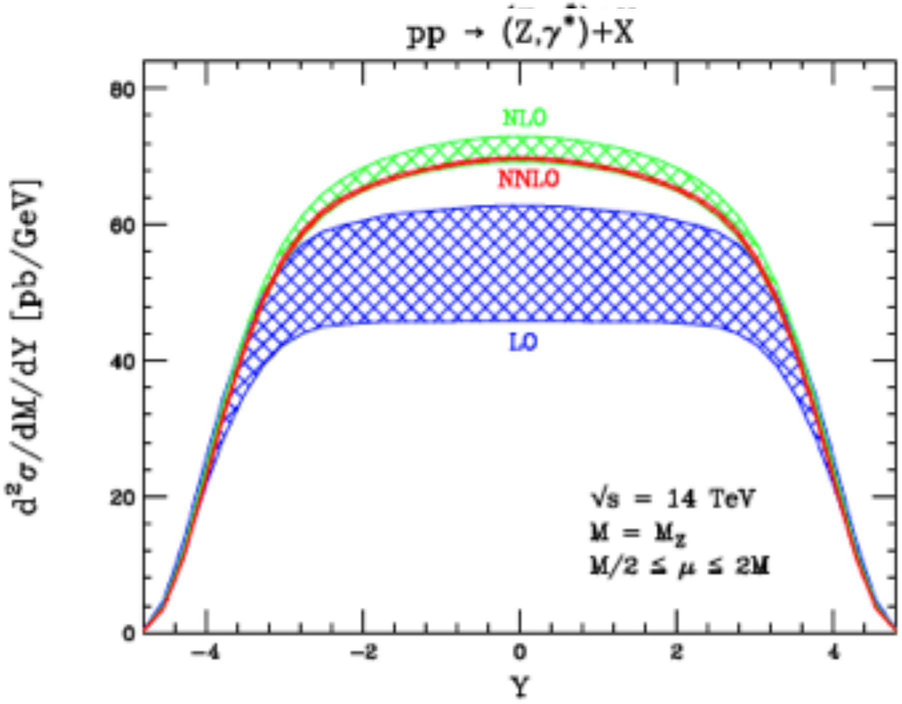
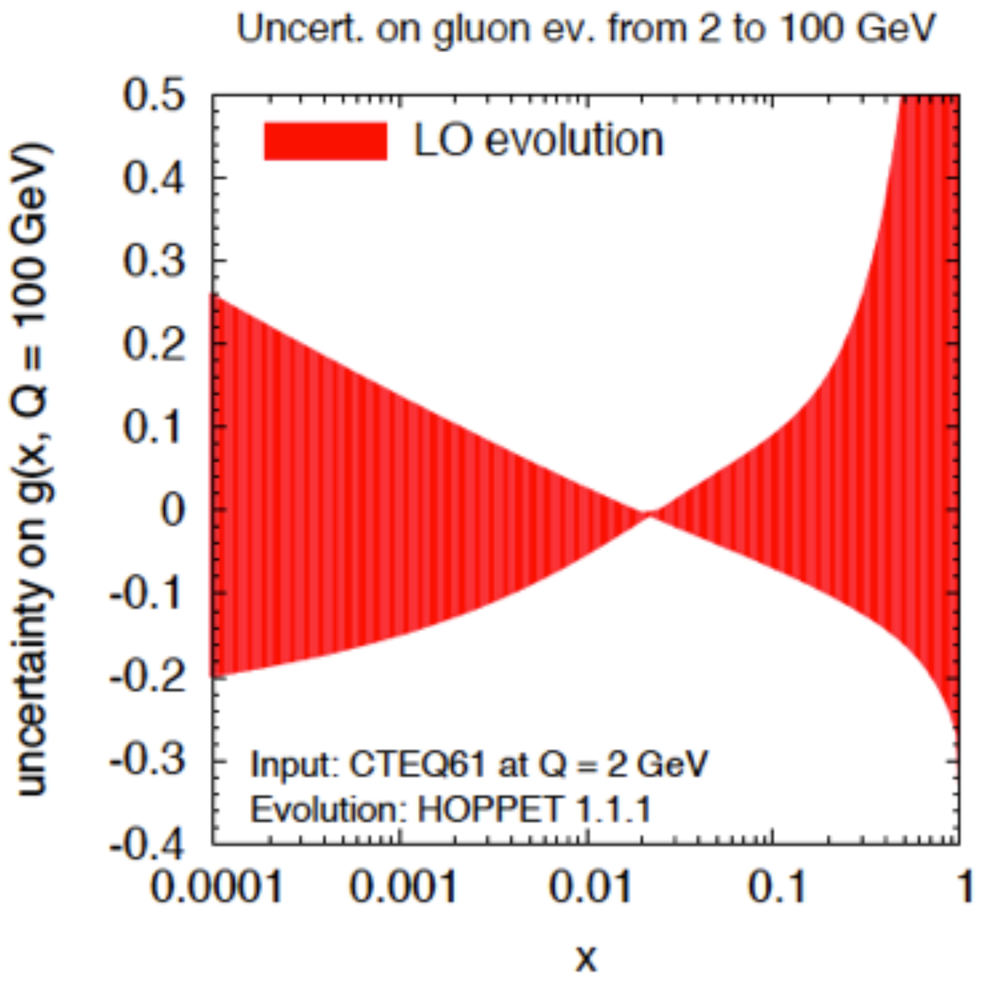


**perturbative accuracy of O(percent) achieved**

# changing scales in DGLAP evolution

estimate by G. Salam: vary the scale of $\alpha_s$ in the DGLAP kernel



- about 30% in LO

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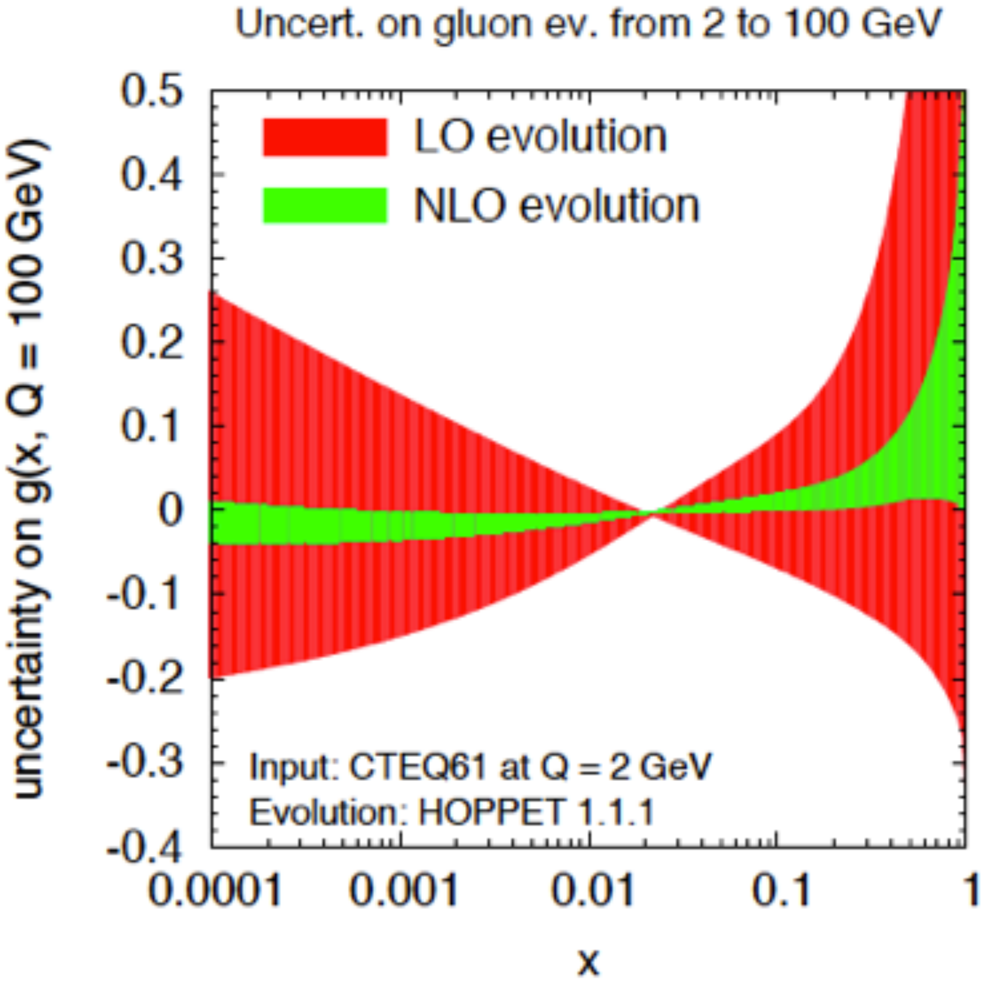

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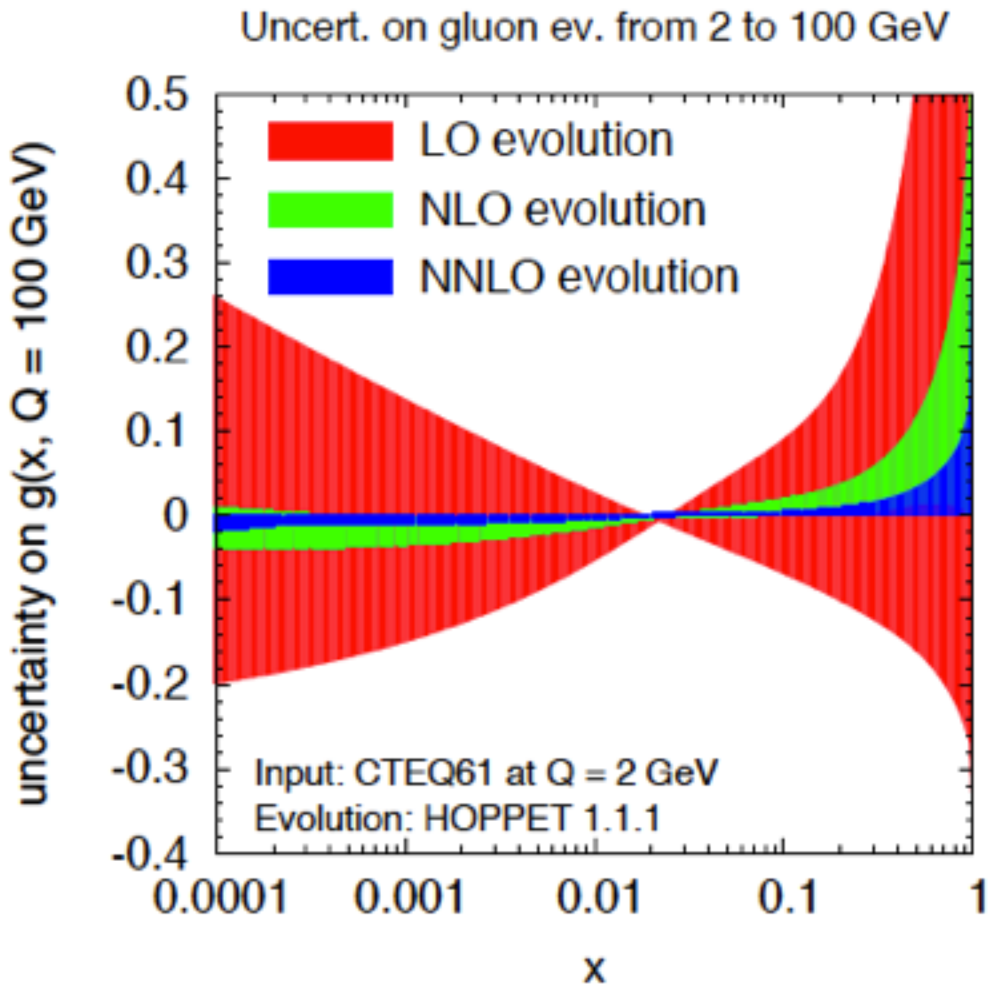

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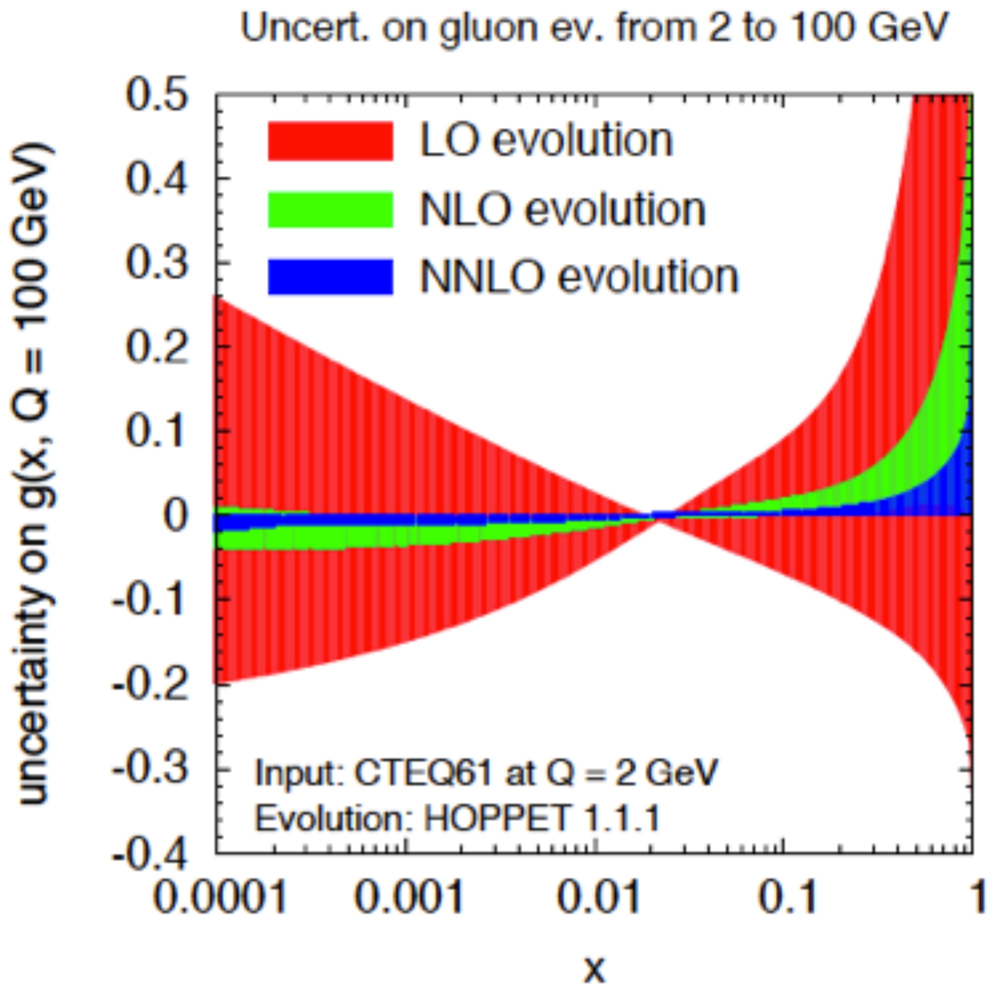

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  which is about the precision of the HERA DIS data

# Anatomy of a Global QCD Analysis

# how to determine PDFs from data?

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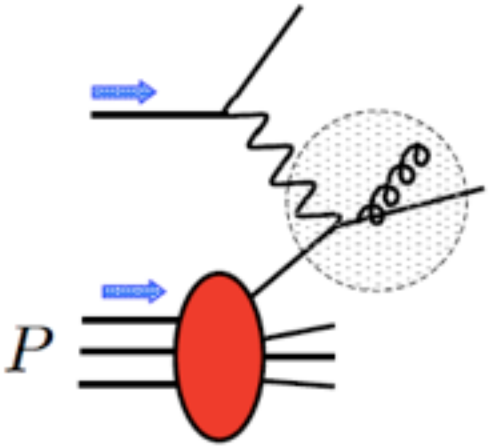
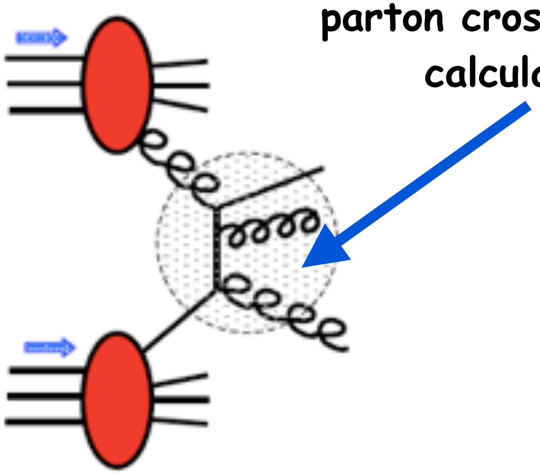


DIS

hard scale Q



parton cross section calculable

hadron-hadron

hard scale $p_T$

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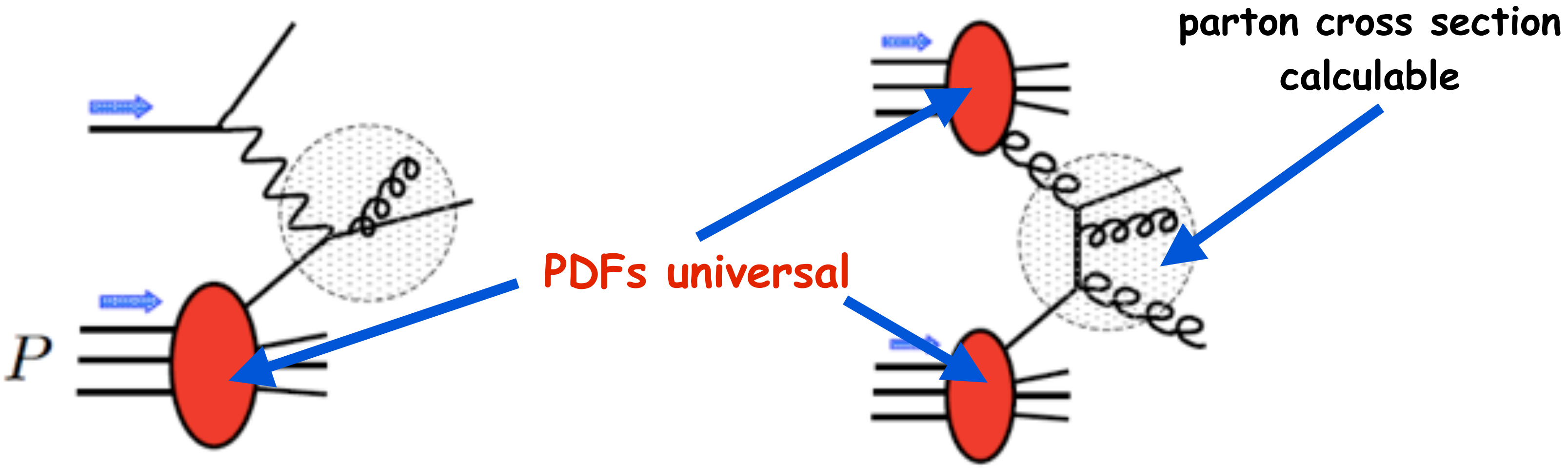

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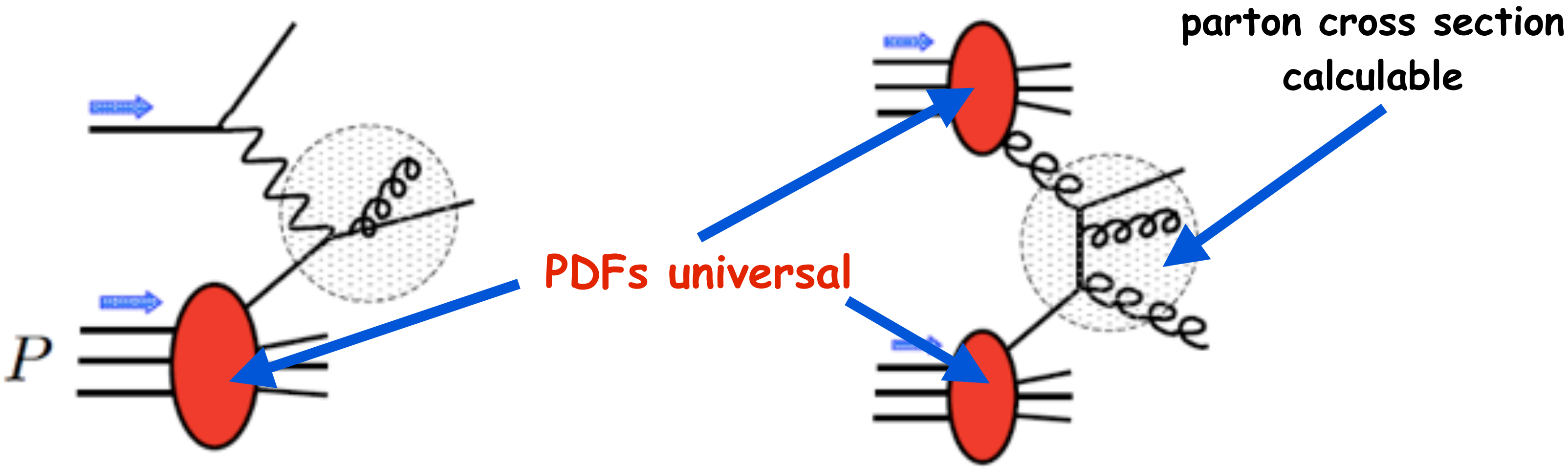

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DIS

hard scale Q

hadron-hadron

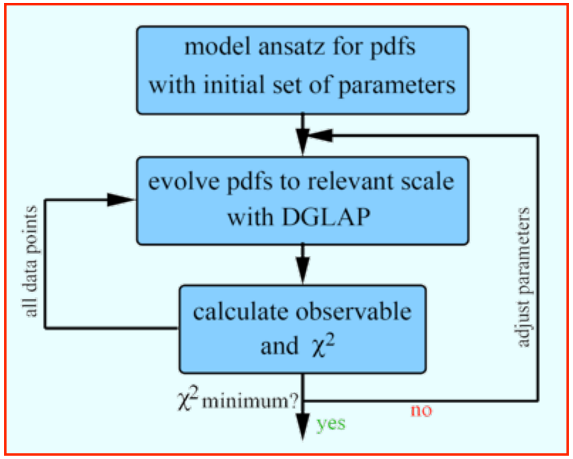
hard scale $p_T$

**task:** extract PDFs and their uncertainties (assume factorization)

- all processes tied together: universality of pdfs & $Q^2$ - evolution
- each reaction provides insights into different aspects and kinematics
- need at least NLO accuracy for quantitative analyses
- information on PDFs "hidden" inside complicated (multi-)convolutions

# anatomy of global PDF analyses

**obtain PDFs**
**through global $\chi^2$ optimization**

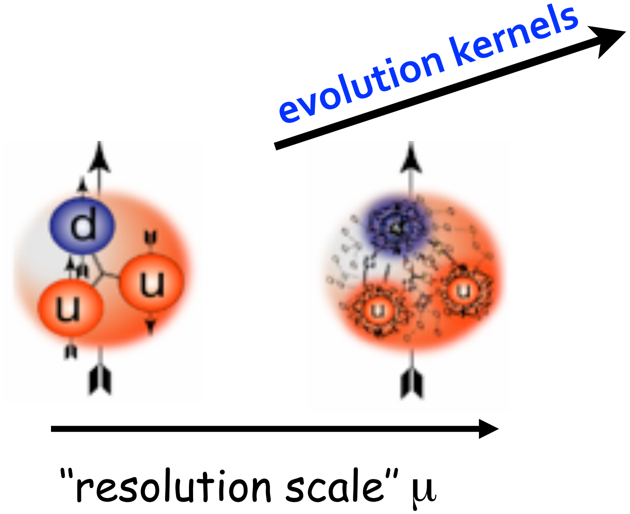


set of **optimum parameters**
for *assumed* functional form

**computational challenge:**

- up to O(20-30) parameters
- many sources of uncertainties
- very time-consuming NLO expressions

# anatomy of global QCD analyses



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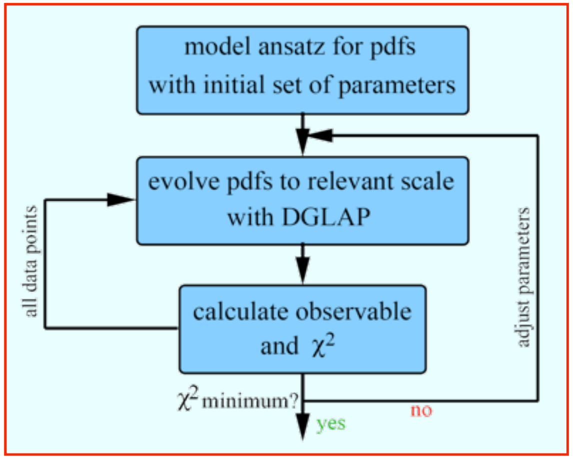

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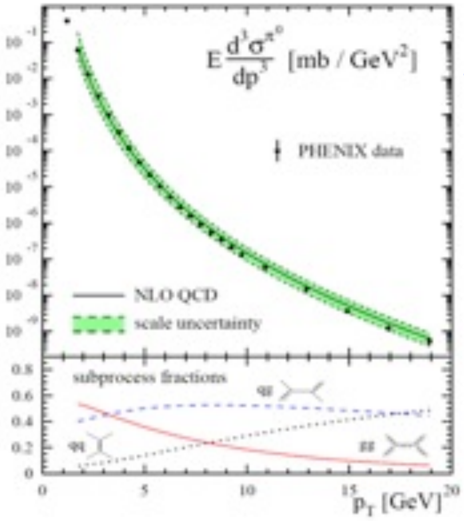

cross sections at NLO

**obtain PDFs**
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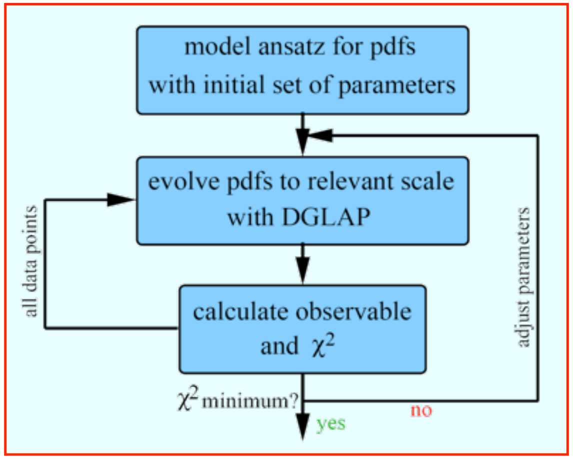


model ansatz for pdfs
with initial set of parameters

evolve pdfs to relevant scale
with DGLAP

calculate observable
and $\chi^2$

all data points

adjust parameters

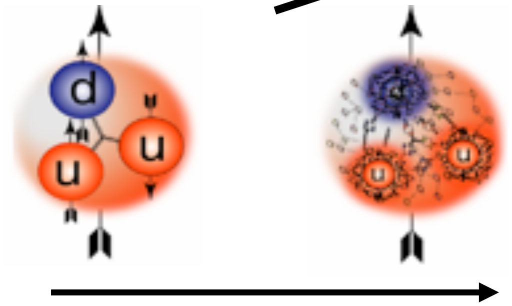
$\chi^2$ minimum?

yes

no

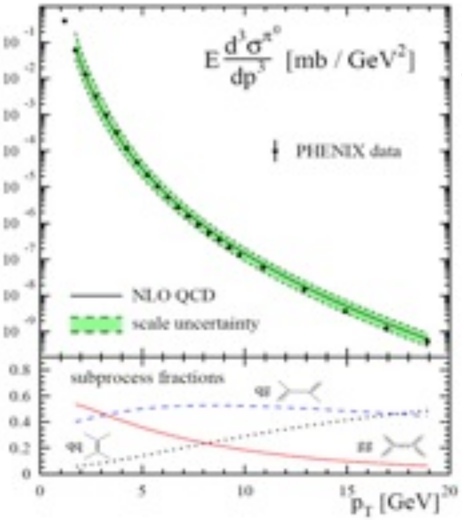
set of **optimum parameters**
for *assumed* functional form

evolution kernels



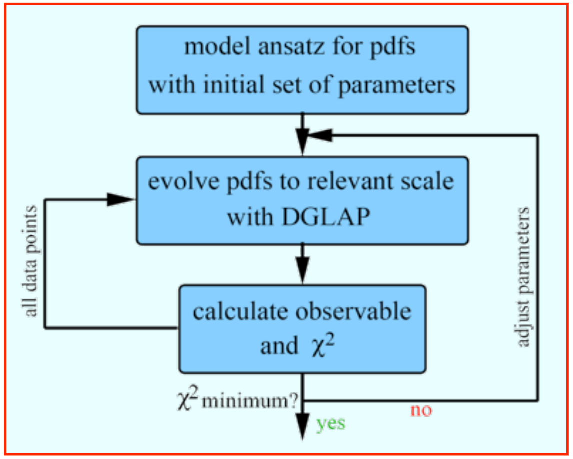
"resolution scale" $\mu$

# anatomy of global QCD analyses



**cross sections at NLO**

**obtain PDFs**
**through global χ² optimization**



set of **optimum parameters**
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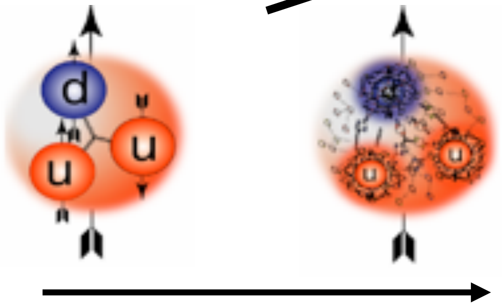
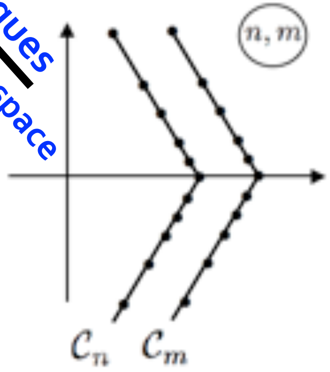
**evolution kernels**

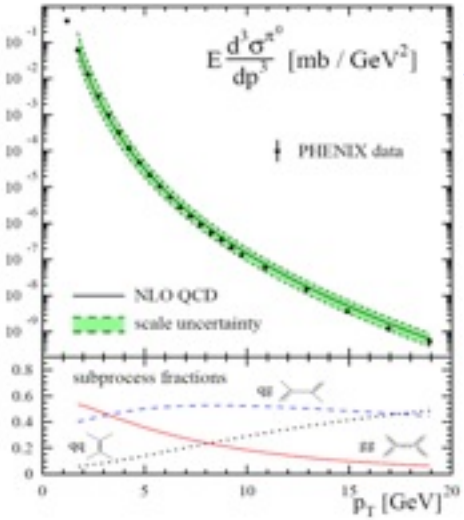


“resolution scale” μ

**novel techniques**
**e.g. in complex Mellin space**



# anatomy of global QCD analyses



cross sections at NLO

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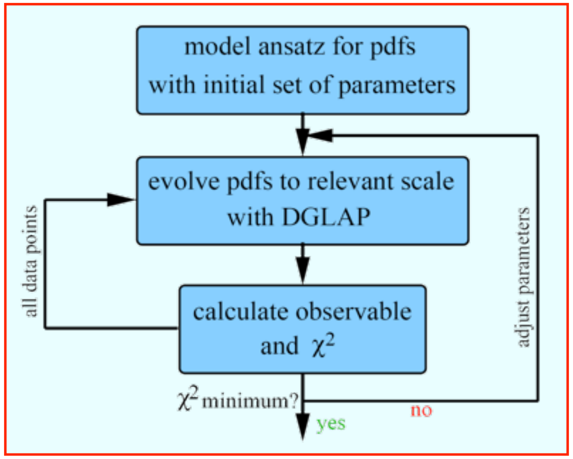
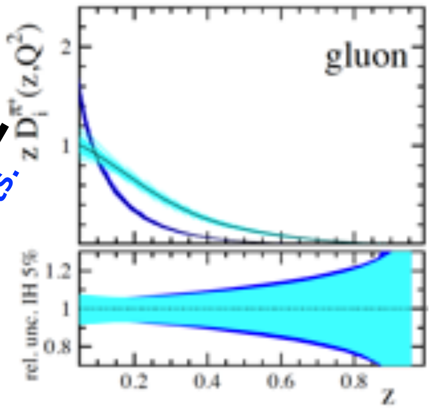
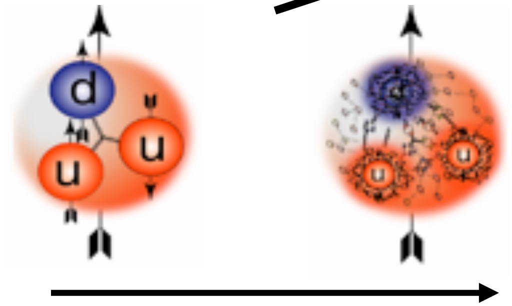


non-pert. inputs
e.g. frag. fcts.



evolution kernels



“resolution scale” μ

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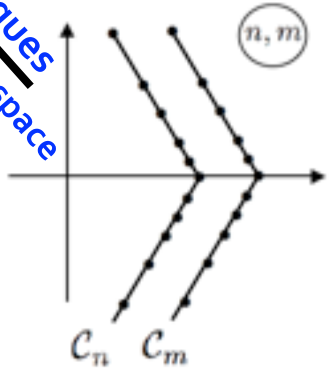

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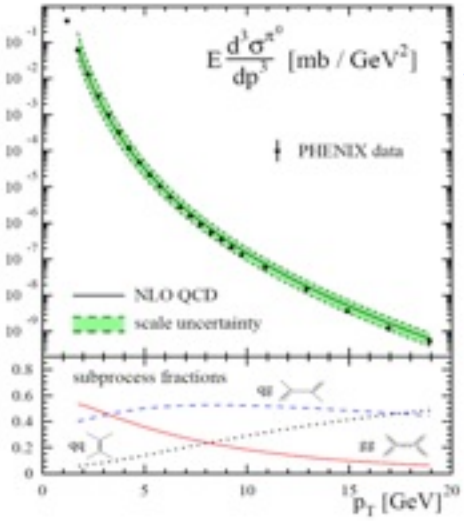

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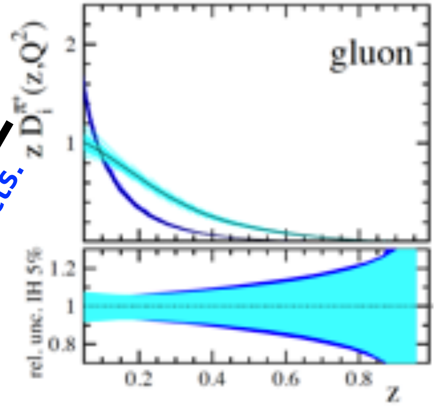

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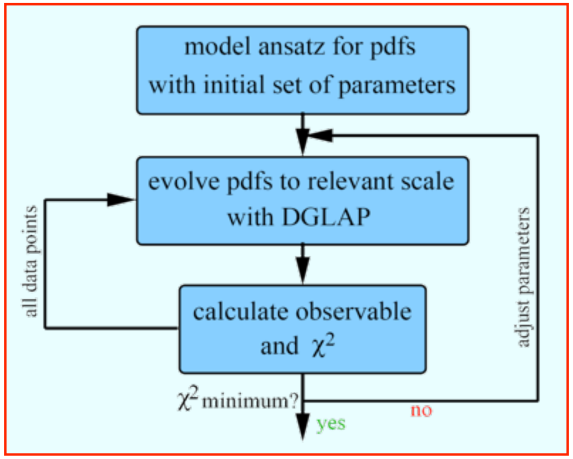

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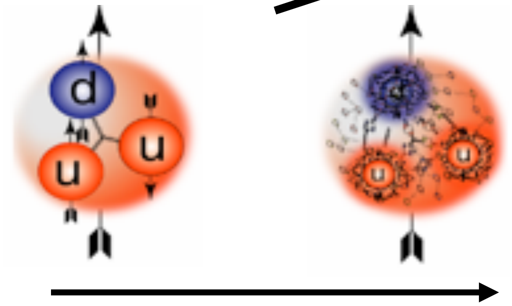
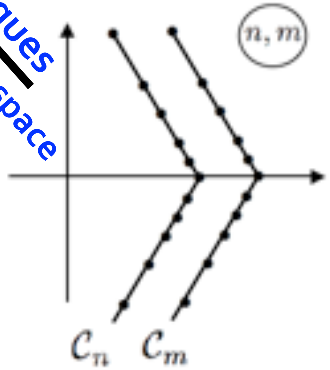


"resolution scale" μ

plus a prescription to
estimate & propagate
uncertainties

novel techniques
e.g. in complex Mellin space



# global analysis: computational challenge

- one has to deal with **O(2800) data points** from many processes and experiments
- need to determine **O(20-30) parameters** describing PDFs at $\mu_0$
- NLO expressions often very complicated → computing time becomes excessive
  → develop **sophisticated algorithms & techniques**, e.g., based on Mellin moments

**Kosower; Vogt; Vogelsang, MS**

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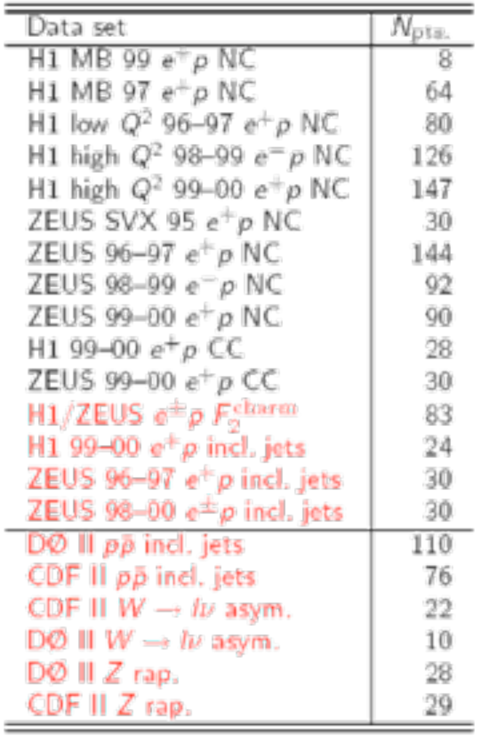
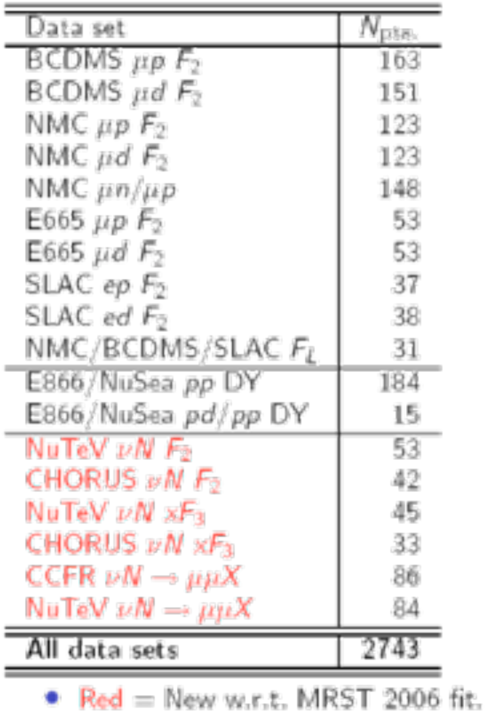
**Kosower; Vogt; Vogelsang, MS**

**data sets & (x,$Q^2$) coverage used in MSTW fit**

**Martin, Stirling, Thorne, Watt, arXiv:0901.0002**

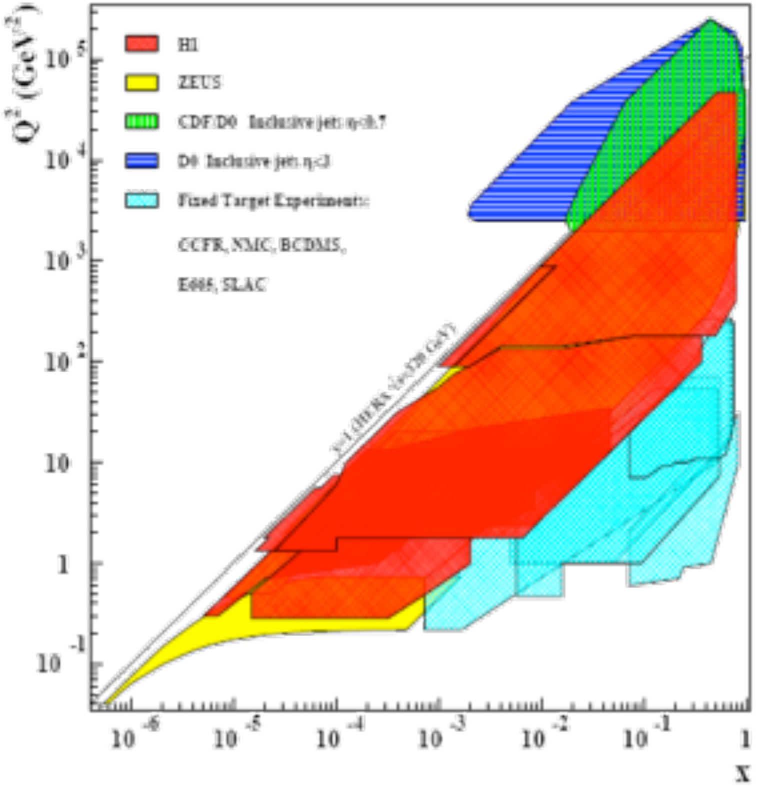


| Data set | $N_{\rm pts.}$ |
|---|---|
| H1 MB 99 $e^+p$ NC | 8 |
| H1 MB 97 $e^+p$ NC | 64 |
| H1 low $Q^2$ 96–97 $e^+p$ NC | 80 |
| H1 high $Q^2$ 98–99 $e^-p$ NC | 126 |
| H1 high $Q^2$ 99–00 $e^+p$ NC | 147 |
| ZEUS SVX 95 $e^+p$ NC | 30 |
| ZEUS 96–97 $e^+p$ NC | 144 |
| ZEUS 98–99 $e^-p$ NC | 92 |
| ZEUS 99–00 $e^+p$ NC | 90 |
| H1 99–00 $e^+p$ CC | 28 |
| ZEUS 99–00 $e^+p$ CC | 30 |
| H1/ZEUS $e^\pm p$ $F_2^{\rm charm}$ | 83 |
| H1 99–00 $e^+p$ incl. jets | 24 |
| ZEUS 96–97 $e^+p$ incl. jets | 30 |
| ZEUS 98–00 $e^\pm p$ incl. jets | 30 |
| DØ II $p\bar{p}$ incl. jets | 110 |
| CDF II $p\bar{p}$ incl. jets | 76 |
| CDF II $W \to l\nu$ asym. | 22 |
| DØ II $W \to l\nu$ asym. | 10 |
| DØ II $Z$ rap. | 28 |
| CDF II $Z$ rap. | 29 |

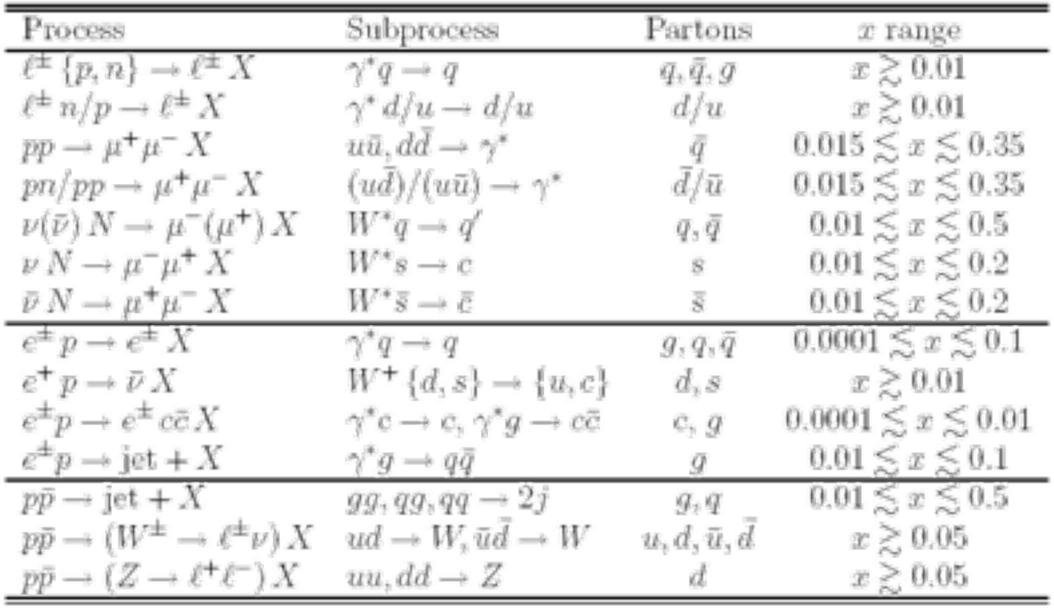


| Data set | $N_{\rm pts.}$ |
|---|---|
| BCDMS $\mu p$ $F_2$ | 163 |
| BCDMS $\mu d$ $F_2$ | 151 |
| NMC $\mu p$ $F_2$ | 123 |
| NMC $\mu d$ $F_2$ | 123 |
| NMC $\mu n/\mu p$ | 148 |
| E665 $\mu p$ $F_2$ | 53 |
| E665 $\mu d$ $F_2$ | 53 |
| SLAC $ep$ $F_2$ | 37 |
| SLAC $ed$ $F_2$ | 38 |
| NMC/BCDMS/SLAC $F_L$ | 31 |
| E866/NuSea $pp$ DY | 184 |
| E866/NuSea $pd/pp$ DY | 15 |
| NuTeV $\nu N$ $F_2$ | 53 |
| CHORUS $\nu N$ $F_2$ | 42 |
| NuTeV $\nu N$ $xF_3$ | 45 |
| CHORUS $\nu N$ $xF_3$ | 33 |
| CCFR $\nu N \to \mu\mu X$ | 86 |
| NuTeV $\nu N \to \mu\mu X$ | 84 |
| All data sets | 2743 |

- Red = New w.r.t. MRST 2006 fit.



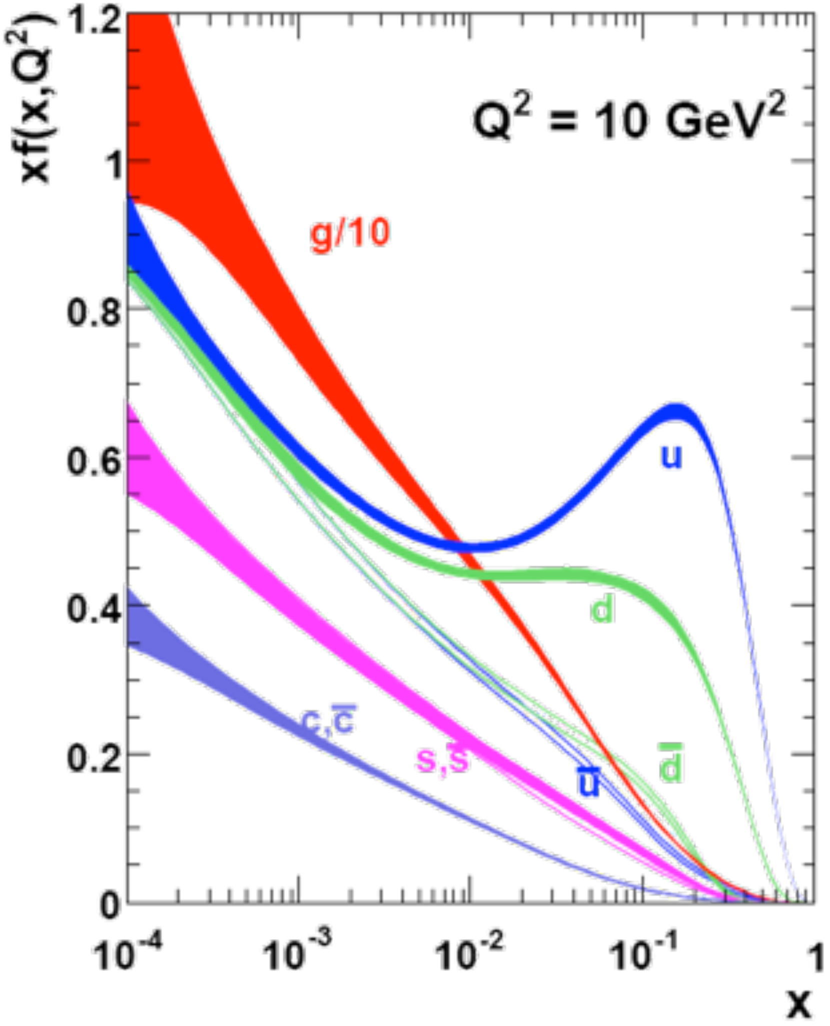
# which data sets determine which partons



| Process | Subprocess | Partons | $x$ range |
|---|---|---|---|
| $\ell^{\pm}\{p,n\} \to \ell^{\pm} X$ | $\gamma^* q \to q$ | $q, \bar{q}, g$ | $x \gtrsim 0.01$ |
| $\ell^{\pm} n/p \to \ell^{\pm} X$ | $\gamma^* d/u \to d/u$ | $d/u$ | $x \gtrsim 0.01$ |
| $pp \to \mu^+\mu^- X$ | $u\bar{u}, d\bar{d} \to \gamma^*$ | $\bar{q}$ | $0.015 \lesssim x \lesssim 0.35$ |
| $pn/pp \to \mu^+\mu^- X$ | $(u\bar{d})/(u\bar{u}) \to \gamma^*$ | $\bar{d}/\bar{u}$ | $0.015 \lesssim x \lesssim 0.35$ |
| $\nu(\bar{\nu}) N \to \mu^-(\mu^+) X$ | $W^* q \to q'$ | $q, \bar{q}$ | $0.01 \lesssim x \lesssim 0.5$ |
| $\nu N \to \mu^-\mu^+ X$ | $W^* s \to c$ | $s$ | $0.01 \lesssim x \lesssim 0.2$ |
| $\bar{\nu} N \to \mu^+\mu^- X$ | $W^* \bar{s} \to \bar{c}$ | $\bar{s}$ | $0.01 \lesssim x \lesssim 0.2$ |
| $e^{\pm} p \to e^{\pm} X$ | $\gamma^* q \to q$ | $g, q, \bar{q}$ | $0.0001 \lesssim x \lesssim 0.1$ |
| $e^+ p \to \bar{\nu} X$ | $W^+\{d,s\} \to \{u,c\}$ | $d, s$ | $x \gtrsim 0.01$ |
| $e^{\pm} p \to e^{\pm} c\bar{c} X$ | $\gamma^* c \to c,\ \gamma^* g \to c\bar{c}$ | $c, g$ | $0.0001 \lesssim x \lesssim 0.01$ |
| $e^{\pm} p \to \text{jet} + X$ | $\gamma^* g \to q\bar{q}$ | $g$ | $0.01 \lesssim x \lesssim 0.1$ |
| $p\bar{p} \to \text{jet} + X$ | $gg, qg, qq \to 2j$ | $g, q$ | $0.01 \lesssim x \lesssim 0.5$ |
| $p\bar{p} \to (W^{\pm} \to \ell^{\pm}\nu) X$ | $ud \to W, \bar{u}\bar{d} \to W$ | $u, d, \bar{u}, \bar{d}$ | $x \gtrsim 0.05$ |
| $p\bar{p} \to (Z \to \ell^+\ell^-) X$ | $uu, dd \to Z$ | $d$ | $x \gtrsim 0.05$ |

Martin, Stirling, Thorne, Watt, arXiv:0901.0002

NLO fit, 68% C.L.



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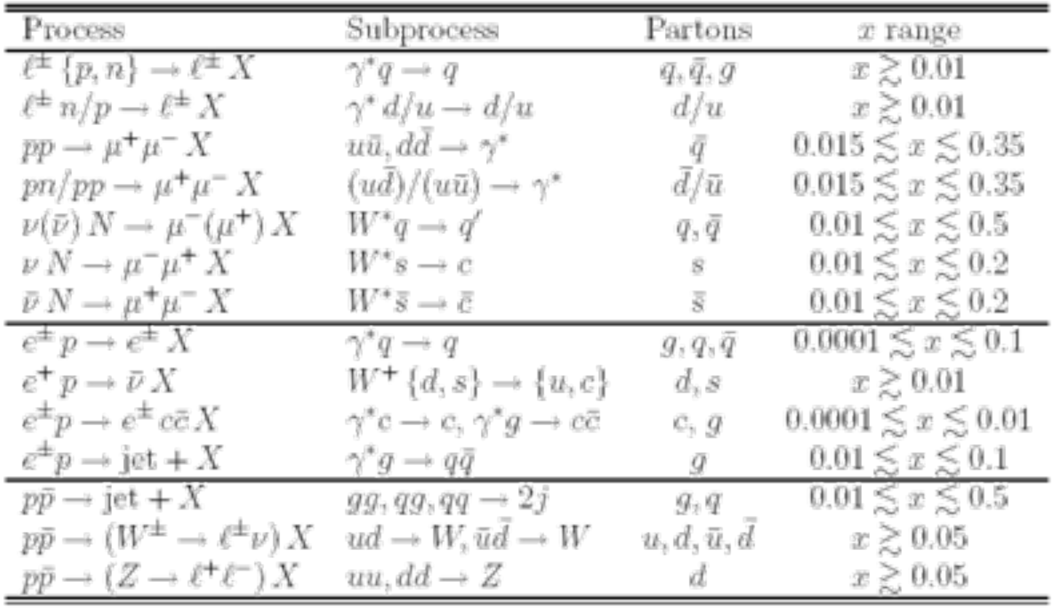

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| $e^+ p \to \bar{\nu} X$ | $W^+\{d,s\} \to \{u,c\}$ | $d, s$ | $x \gtrsim 0.01$ |
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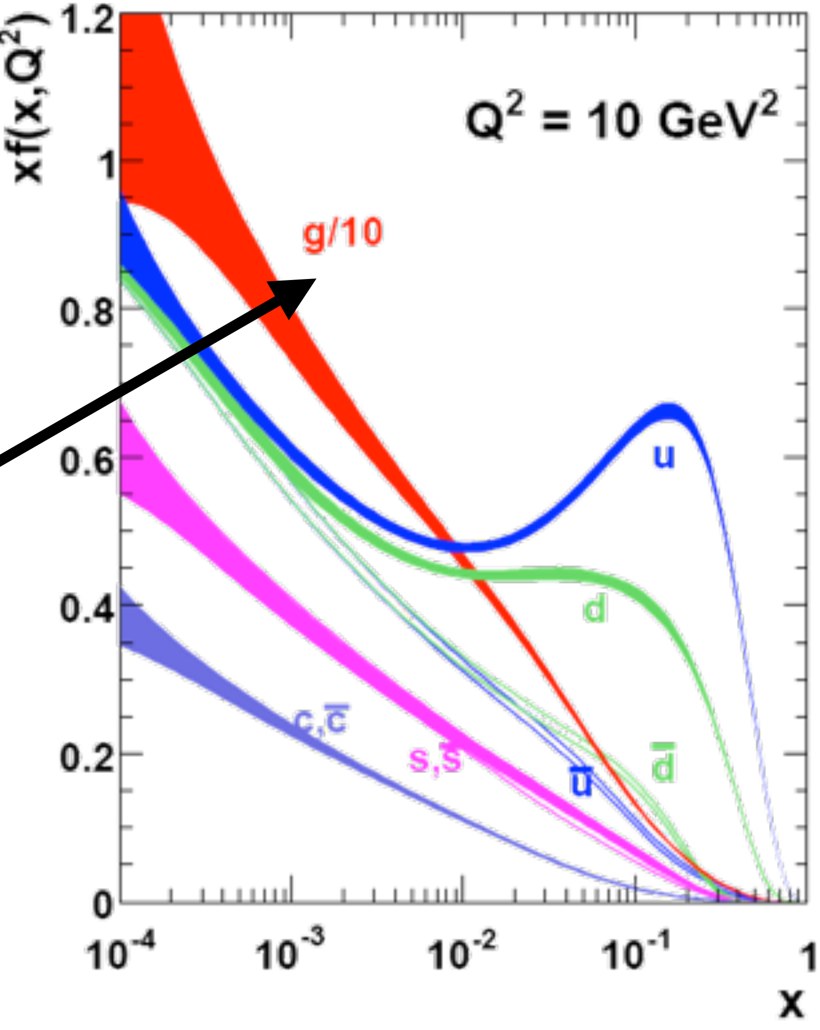


- notice the huge gluon distribution
- quality of the fit:

$\chi^2$/ #data pts.

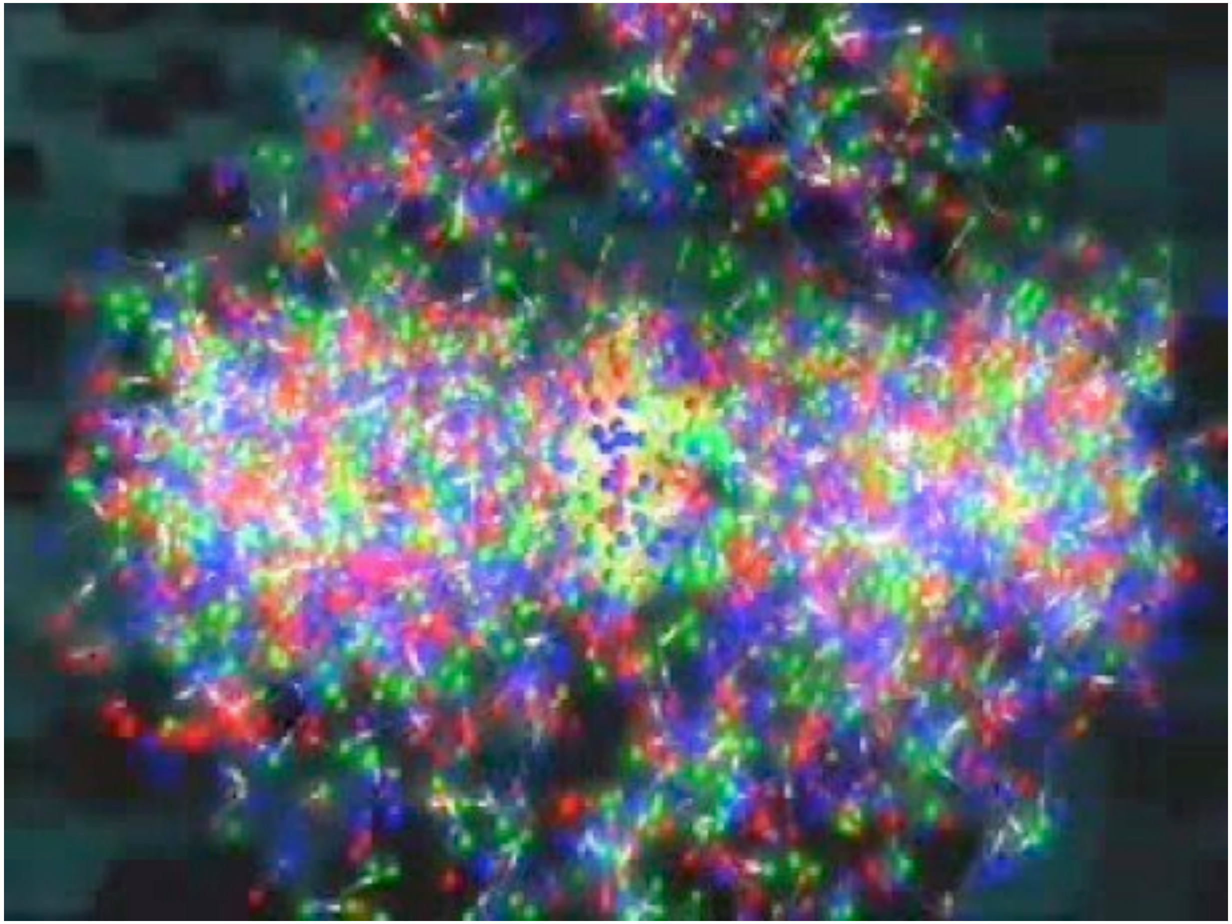
- 2543/2699 **NLO**
- 3066/2598 **LO**

**interplay of many data sets crucial**


PARTON
Drive carefully
Burial place of
James Clerk Maxwell

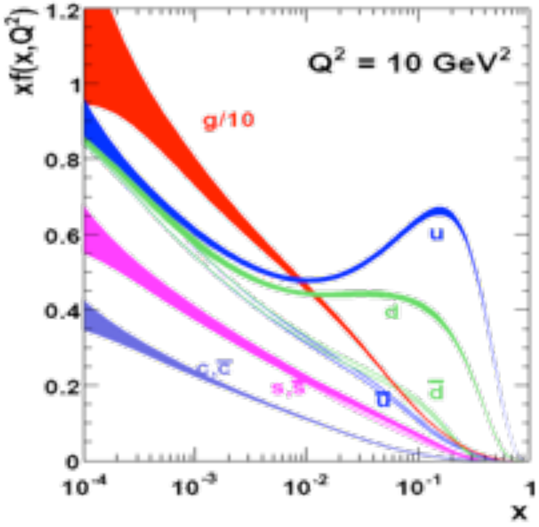
from R.D. Ball

# 3



# when there is not enough room: gluons at small x

# what drives the growth of the gluon density



observe that only 2 splitting fcts are singular at small x

$$P_{gq}(x)\big|_{x\to 0} \approx \frac{2C_F}{x} \qquad P_{gg}(x)\big|_{x\to 0} \approx \frac{2C_A}{x}$$

-> small x region dominated by gluons

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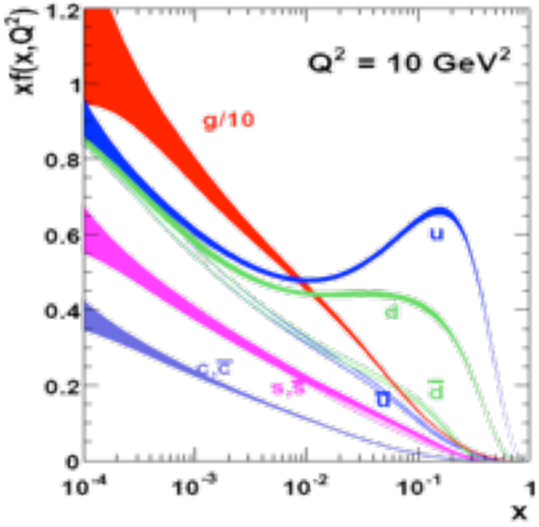

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- write down **“gluon-only” DGLAP equation** **only valid for small x and large Q$^2$**

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\frac{2C_A}{z} g(x/z,\mu^2)$$

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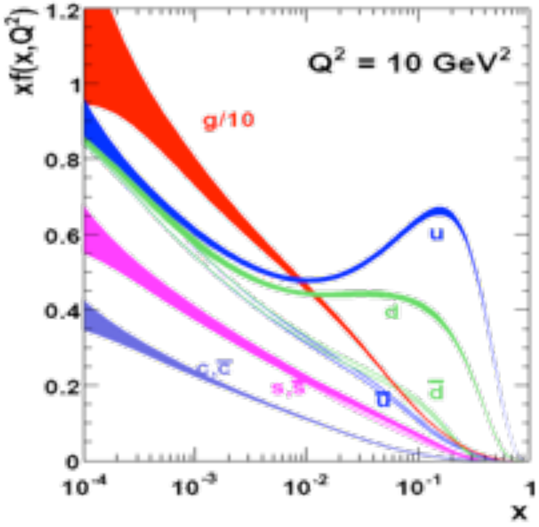

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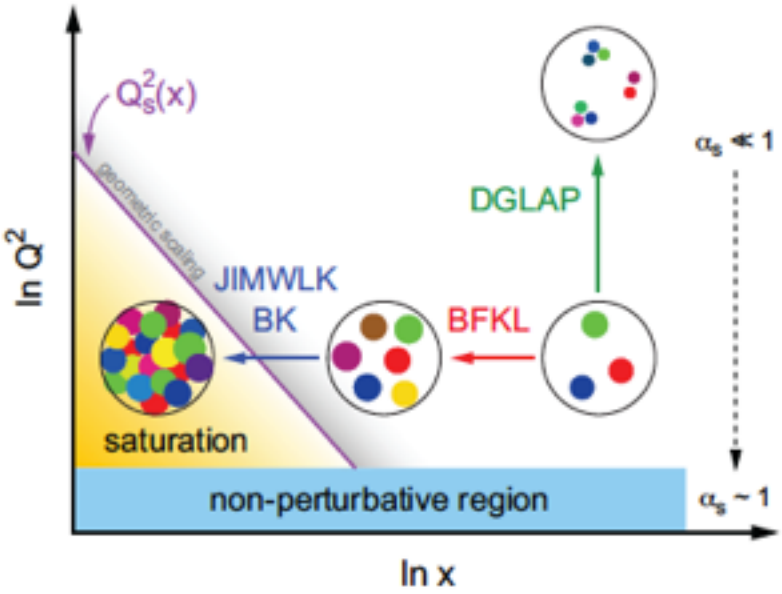
- for fixed coupling this leads to **"double logarithmic approximation"**

$$xg(x,Q^2) \sim \exp\left(2\sqrt{\frac{\alpha_S C_A}{\pi}\log(1/x)\log(Q^2/Q_0^2)}\right)$$

predicts rise that is faster than $\log^a(1/x)$ but slower than $(1/x)^a$

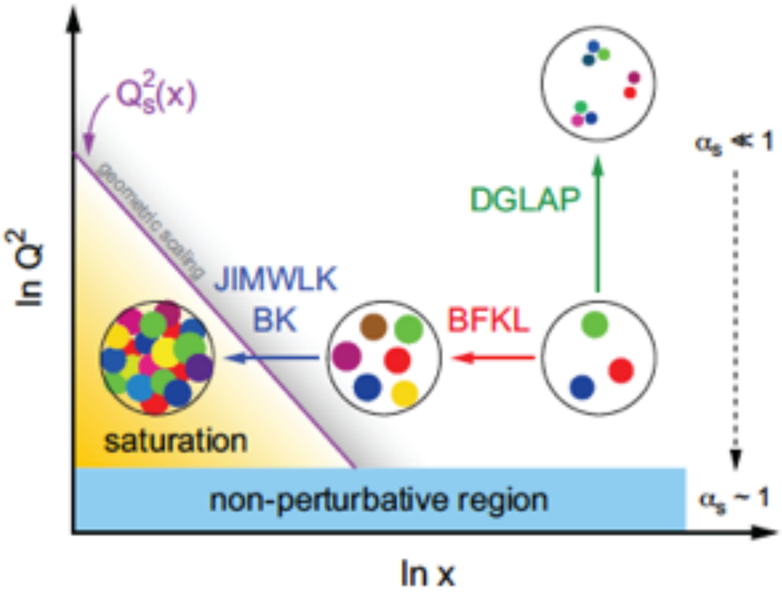
# gluon occupancy



- DGLAP predicts an increase of gluons at small x but proton becomes more dilute as $Q^2$ increases

transverse size of partons ≈ 1/Q

# gluon occupancy



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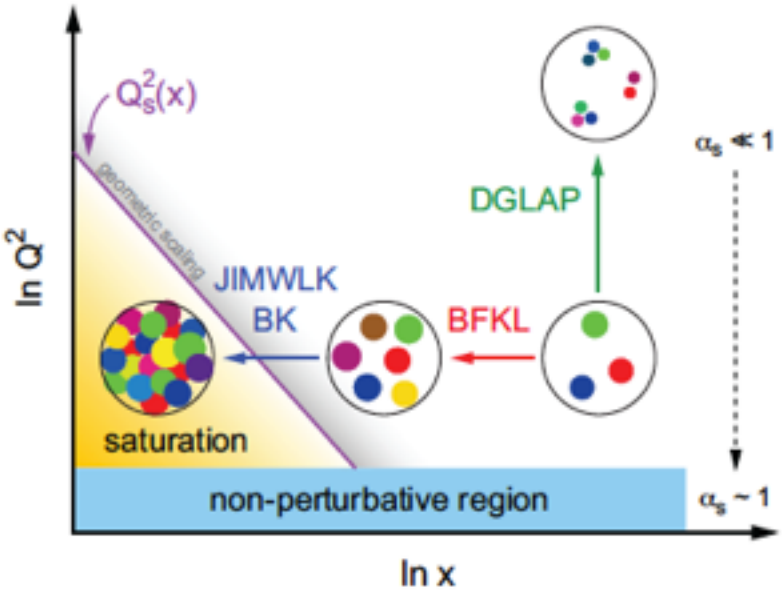

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**"high-energy (Regge) limit of QCD"**

- aim to resum terms ≈ $\alpha_s \log(1/x)$
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- BFKL predicts a power-like growth $xg(x,Q^2) \sim (1/x)^{\alpha_P - 1}$

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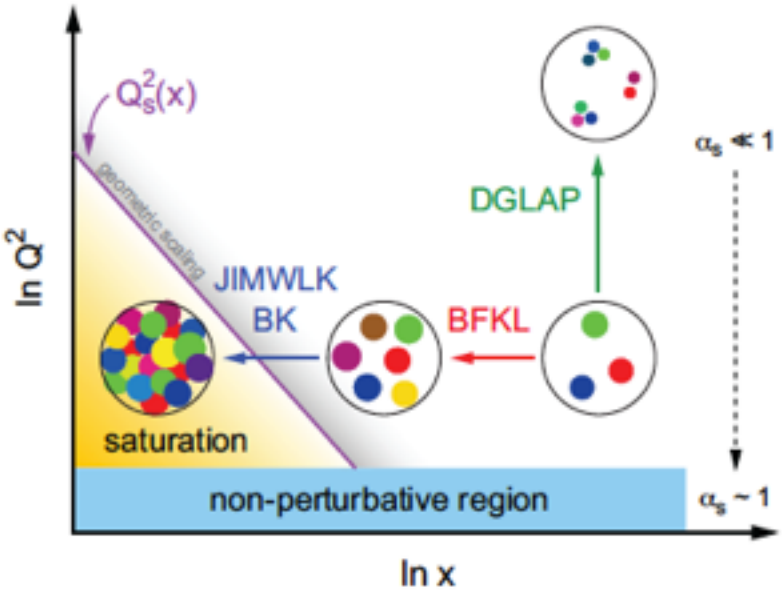

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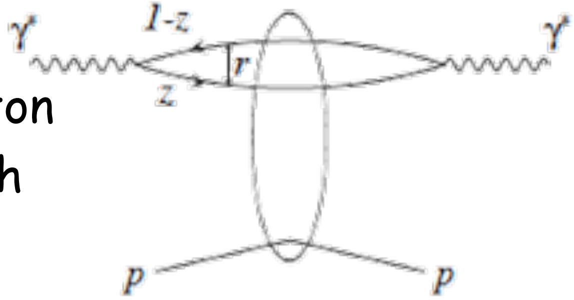
**BIG problem**

- proton quickly fills up with gluons (transverse size now fixed !)
- hadronic cross sections violate $\ln^2 s$ bound (Froissart-Martin) and grow like a power

# color dipole model

make progress by viewing, e.g., DIS from a "different angle"



DIS in the **proton rest frame** can be viewed as the photon splitting into a quark-antiquark pair ("**color dipole**") which scatters off the proton (= "slow" gluon field)

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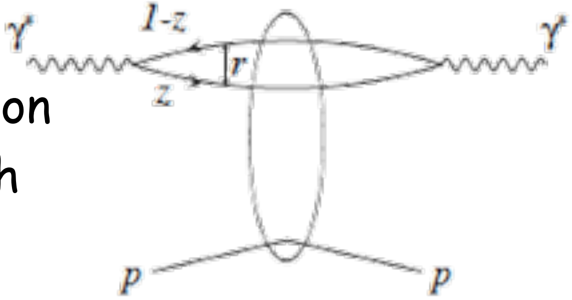

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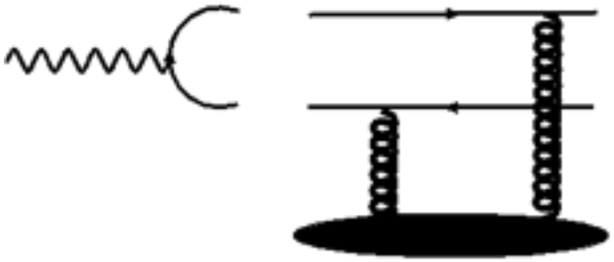

= **probability of photon fluctuating into qq-pair** ⊗ **probability of dipole scattering on the target**

**QED** **QCD**

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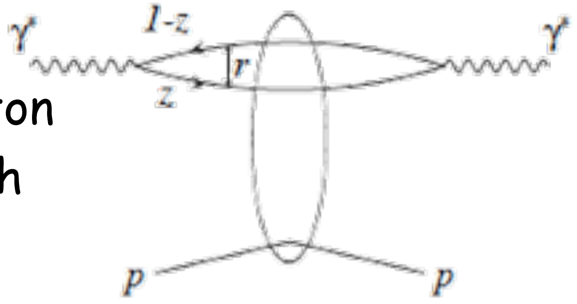

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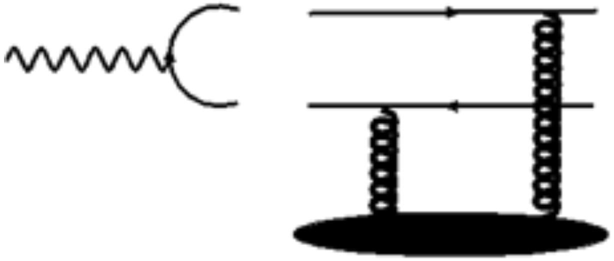

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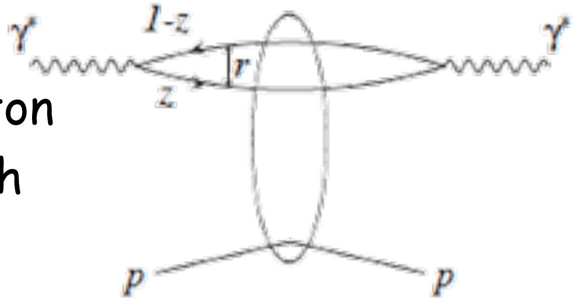

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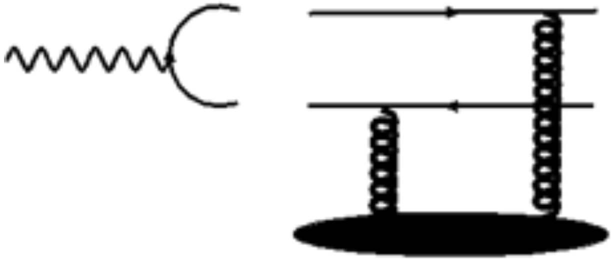

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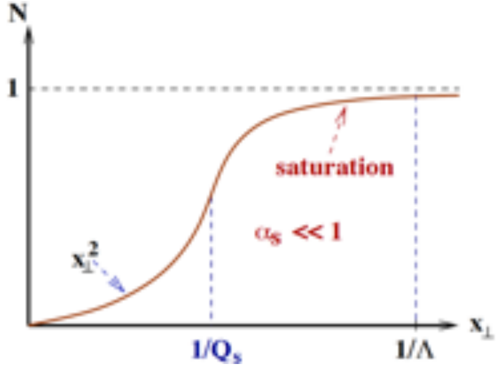
**QED** **QCD**

- introduces **dipole-nucleon scattering amplitude N** as fund. building block
- energy dependence of N described by Balitsky-Kovchegov equation
  - **non-linear** -> includes multiple scatterings for unitarization
  - generates saturation scale $Q_s$
  - suited to treat collective phenomena (shadowing, diffration)
  - impact parameter dependence



# 4

## when $N^xLO$ is not enough: all order resummations

# when a N$^x$LO calculation is not good enough

**observation:** fixed N$^x$LO order QCD calculations are not necessarily reliable
this often happens at low energy fixed-target experiments
and can be an issue also at colliders, even the LHC

**reason:** structure of the perturbative series and IR cancellation

at partonic threshold / near exclusive boundary:

- just enough energy to produce, e.g., high-$p_T$ parton
- "inhibited" radiation (general phenomenon for gauge theories)

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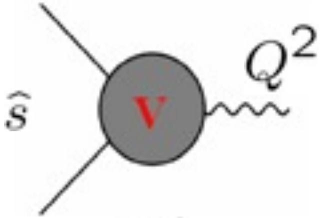
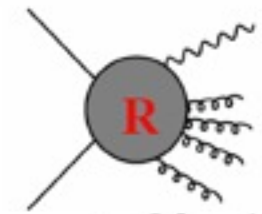
**simple example:**
**Drell-Yan process**





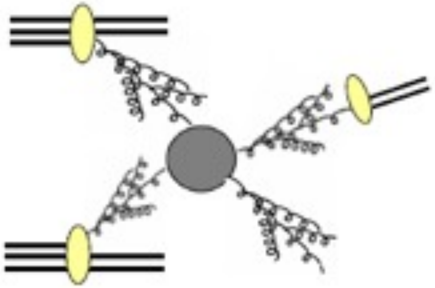
$$z \equiv \frac{Q^2}{\hat{s}} = 1$$

$$\propto \alpha_s^k \, \frac{\ln^{2k-1}(1-z)}{1-z}$$

"imbalance" of real and virtual contributions: **IR cancellation leaves large log's**

# all order structure of partonic cross sections

**let's consider pp scattering:**



logarithms related to partonic threshold $\hat{x}_T = \frac{2p_T}{\sqrt{\hat{s}}} \rightarrow 1$

general structure of partonic cross sections at the k<sup>th</sup> order:

$$p_T^3 \frac{d\hat{\sigma}_{ab}}{dp_T} = p_T^3 \frac{d\hat{\sigma}_{ab}^{\mathrm{Born}}}{dp_T} \Big[ 1 + \underbrace{\mathcal{A}_1\, \alpha_s \ln^2\left(1-\hat{x}_T^2\right) + \mathcal{B}_1\, \alpha_s \ln\left(1-\hat{x}_T^2\right)}_{\text{NLO}}$$

$$+ \ldots + \underbrace{\mathcal{A}_k\, \alpha_s^k \ln^{2k}\left(1-\hat{x}_T^2\right)}_{\text{"threshold logarithms"}} + \ldots \Big] + \ldots$$

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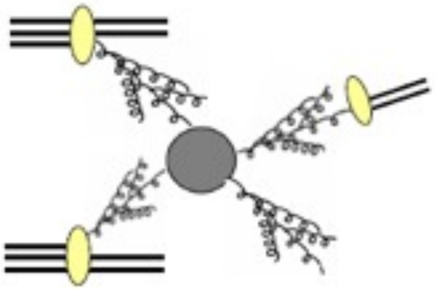

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$$+ \ldots + \mathcal{A}_k\, \alpha_s^k \ln^{2k}\left(1-\hat{x}_T^2\right) + \ldots \Big] + \ldots$$

"threshold logarithms"

**where relevant?** … convolution with steeply falling parton luminosity $L_{ab}$:

$$d\sigma \propto \sum_{a,b} \int_\tau^1 \frac{dz}{z} \mathcal{L}_{ab}\left(\frac{\tau}{z}\right) d\hat{\sigma}_{ab}(z)$$

z = 1 emphasized, in particular as τ → 1

large at small τ/z

→ important for fixed target phenomenology: threshold region more relevant (large $\tau$ )

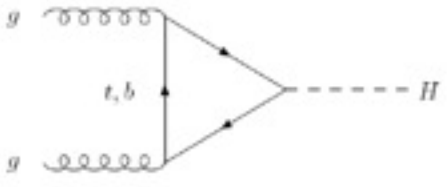
# resummations – how are they done

$\alpha_s^k \ln^{2k}(1-\hat{x}_T^2)$ **may spoil perturbative series - unless taken into account to all orders**

**resummation** of such terms has reached a high level of sophistication

Sterman; Catani, Trentadue; Laenen, Oderda, Sterman;
Catani et al.; Sterman, Vogelsang; Kidonakis, Owens; ...

- worked out for most processes of interest at least to NLL
- well defined class of higher-order corrections
- often of much phenomenological relevance even for high mass particle production at the LHC



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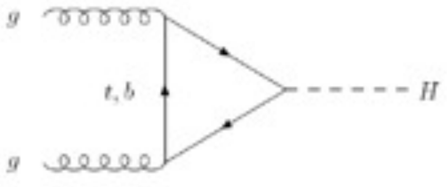

resummation (= exponentiation) occurs when "right" moments are taken:

Mellin moments for threshold logs $\alpha_s^k \ln^{2k}(1-\hat{x}_T^2) \rightarrow \alpha_s^k \ln^{2k}(N)$

- fixed order calculations needed to determine "coefficients"
- the more orders are known, the more subleading logs can be resummed

# resummations – terminology

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**Fixed order calculation**

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**LO**

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**Fixed order calculation**

**LO**

**NLO** $\alpha_s L^2 \quad \alpha_s L \quad \alpha_s \quad + \ldots$

# resummations – terminology

**Fixed order calculation**

| | | | | | |
|---|---|---|---|---|---|
| **LO** | | | | | |
| **NLO** | $\alpha_s \, L^2$ | $\alpha_s \, L$ | $\alpha_s$ | | $+ \ldots$ |
| **NNLO** | $\alpha_s^2 \, L^4$ | $\alpha_s^2 \, L^3$ | $\alpha_s^2 \, L^2$ | $\alpha_s^2 \, L$ | $+ \ldots$ |

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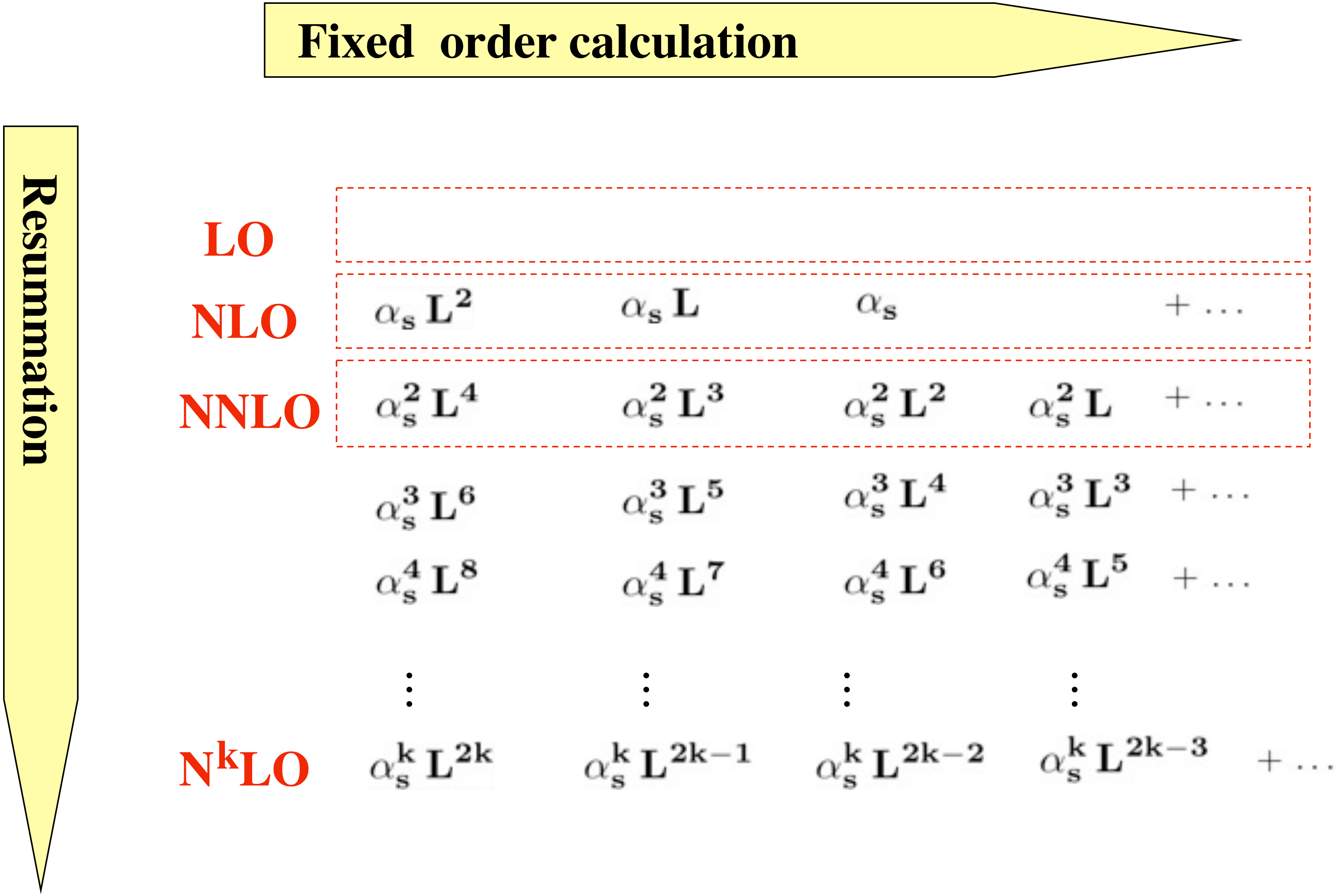

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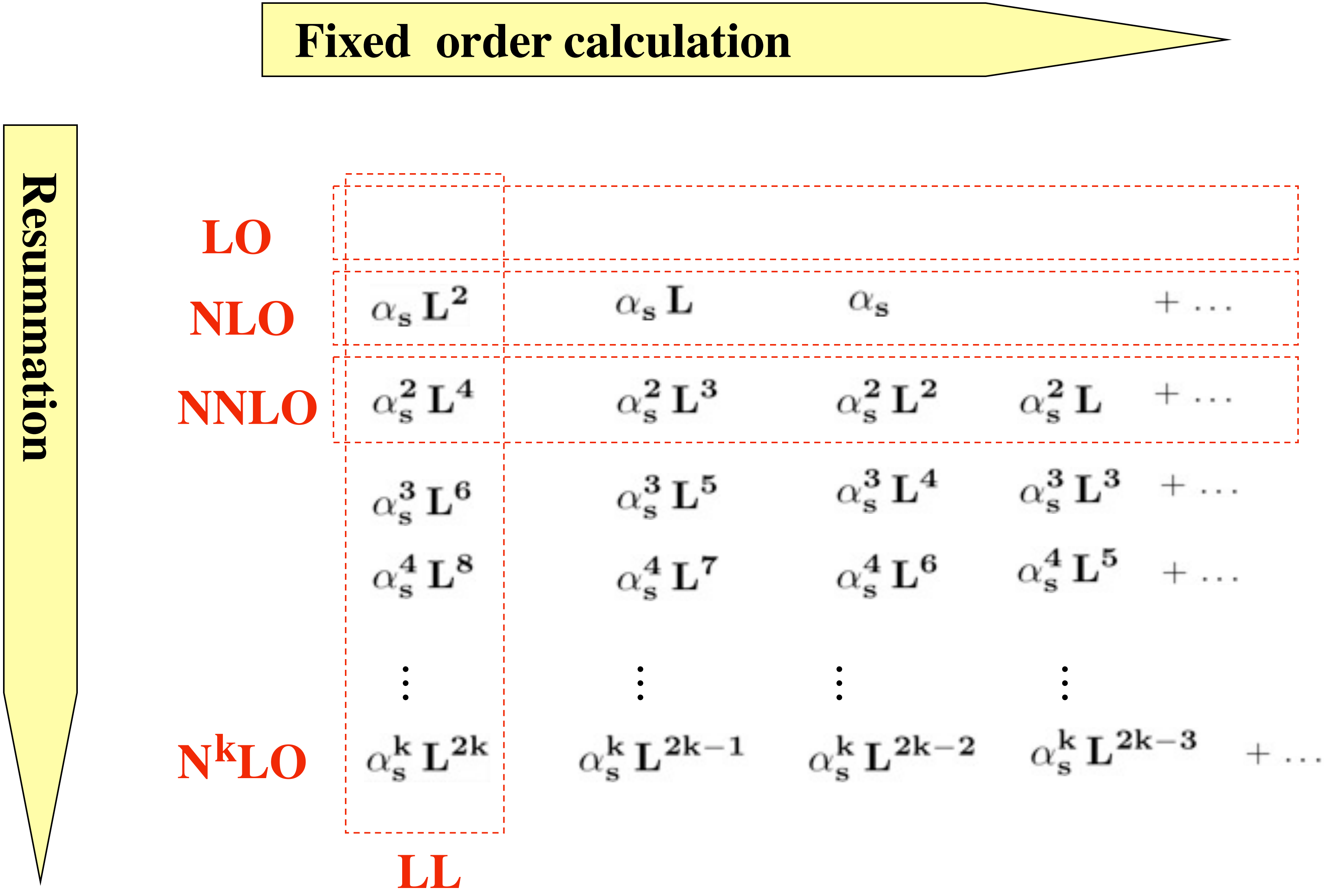

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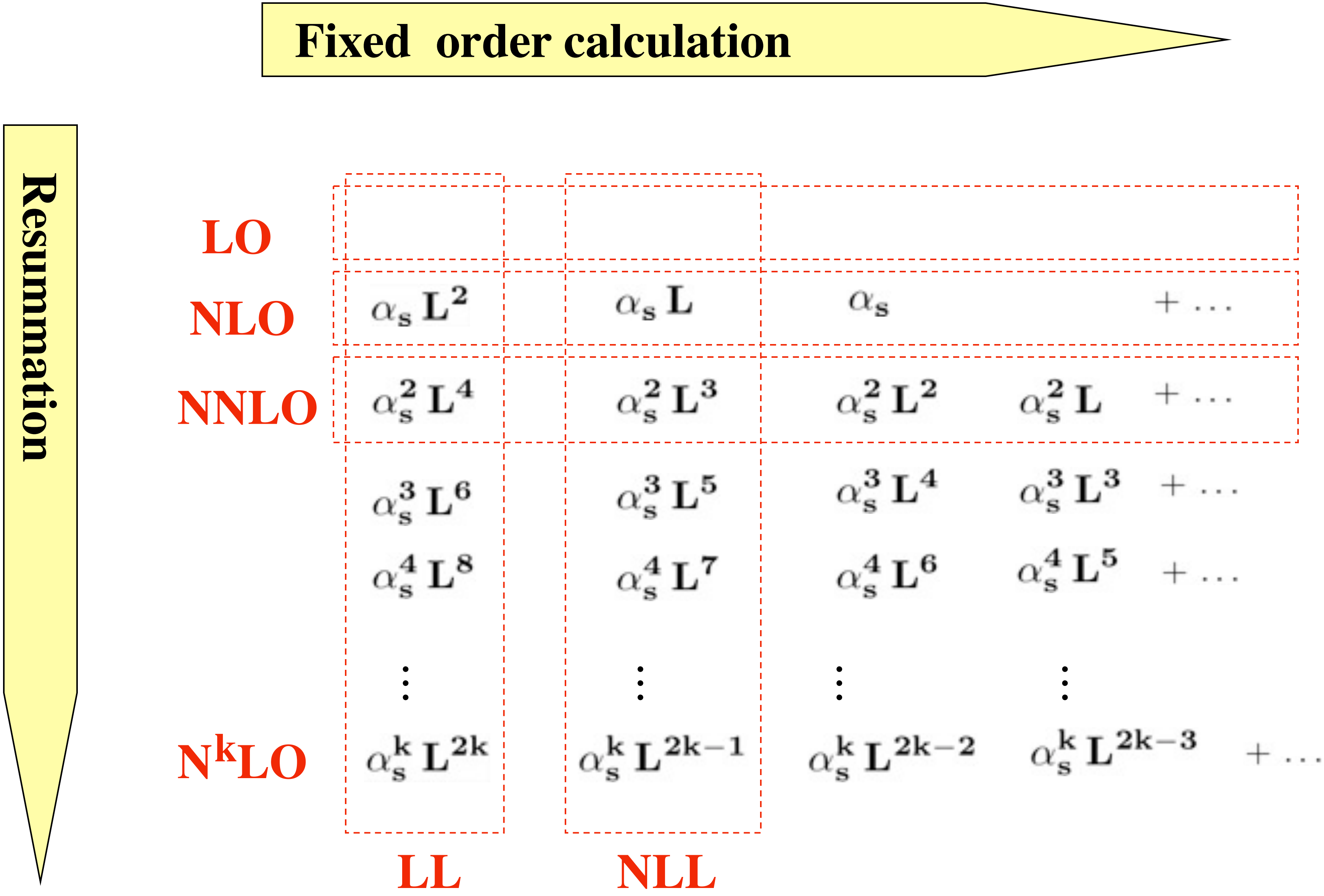

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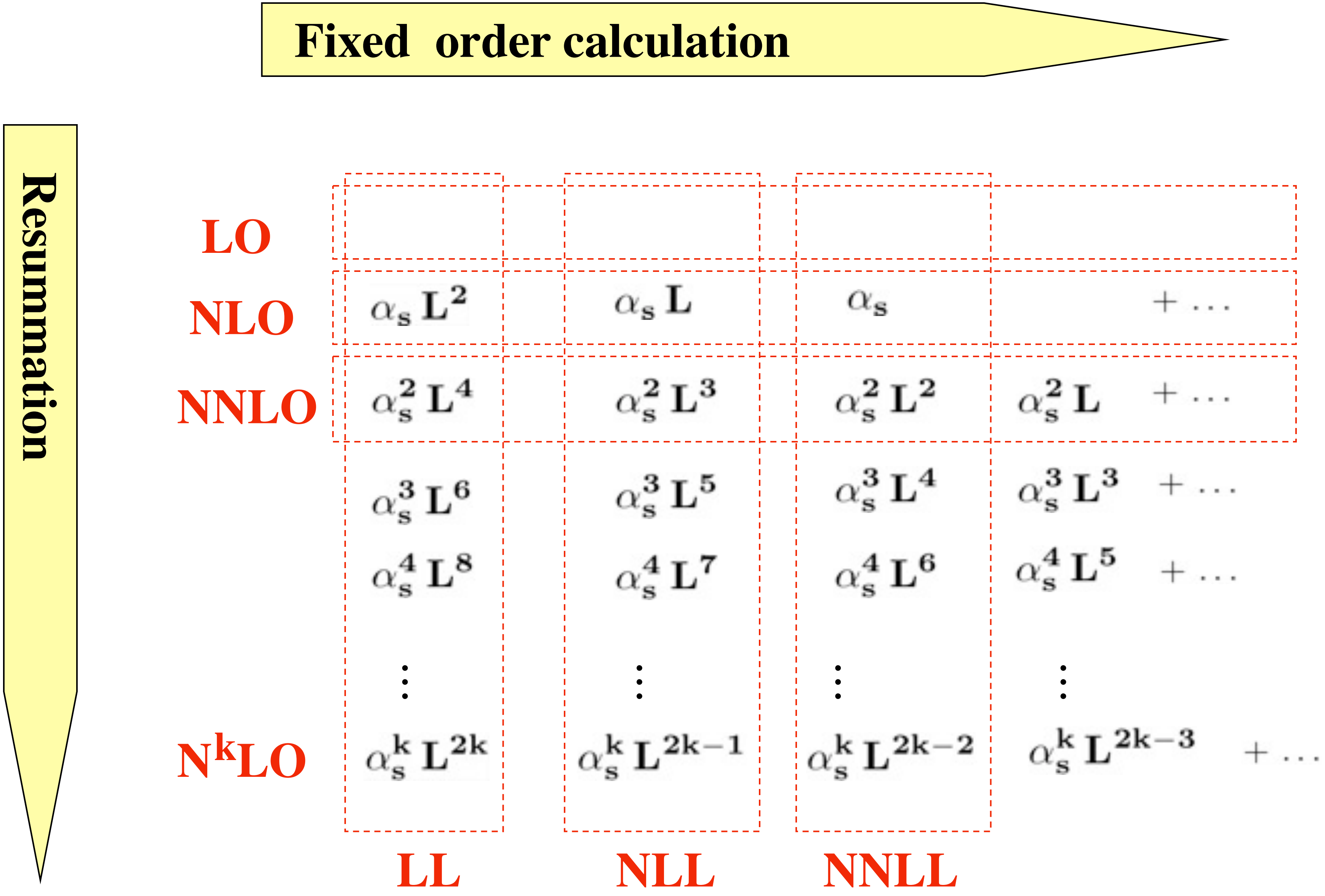

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| | LL | NLL | NNLL | | |

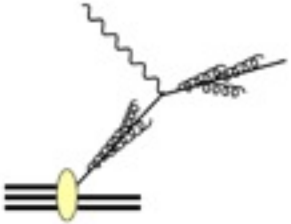
# some leading log exponents

(assuming fixed $\alpha_s$ for simplicity)

**color factors for soft gluon radiation matter:**

unobserved parton Sudakov "suppression"

DIS



$$\exp\left[\frac{C_F\,\alpha_s}{\pi}\ln^2(N)-\frac{C_F\,\alpha_s}{\pi}\frac{1}{2}\ln^2(N)\right]$$

moderate enhancement, unless $x_{Bj}$ large

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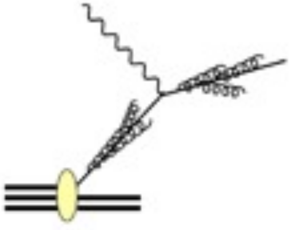

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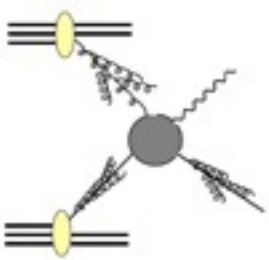

$$q\bar{q} \to \gamma g \qquad \exp\left[\left(C_F + C_F - \frac{1}{2}C_A\right)\frac{\alpha_s}{\pi}\ln^2(N)\right]$$

$$qg \to \gamma q \qquad \exp\left[\left(C_F + C_A - \frac{1}{2}C_F\right)\frac{\alpha_s}{\pi}\ln^2(N)\right]$$

exponents positive ⟶ enhancement

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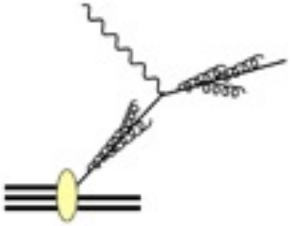

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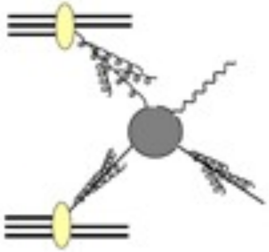

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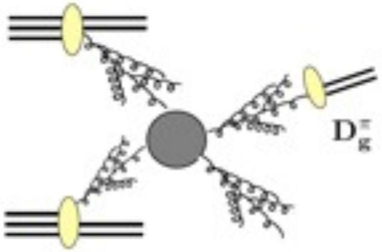
exponents positive ⟶ enhancement

inclusive hadrons



observed partons

unobserved

e.g.
$gg \rightarrow gg$ $$\exp\left[\left(C_A + C_A + C_A - \frac{1}{2}C_A\right)\frac{\alpha_s}{\pi}\ln^2(N)\right]$$

expect much larger enhancement

# resummations: window to non-perturbative regime

**important technical issue:**

resummations are sensitive to strong coupling regime

→ need some "minimal prescription" to avoid Landau pole (where $\alpha_s \rightarrow \infty$)

Catani, Mangano, Nason, Trentadue:

define resummed result such that series is asymptotic
w/o factorial growth associated with power corrections
[achieved by particular choice of Mellin contour]

→ power corrections may be added afterwards if pheno. needed
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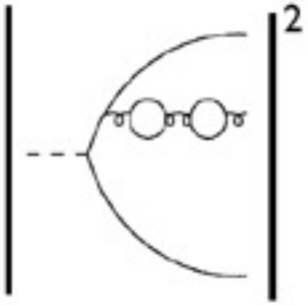
**window to the non-perturbative regime so far little explored**

# “convergence” of an asymptotic series

see, “Renormalons” review by M. Beneke, hep-ph/9807443

suppose we keep calculating
higher and higher orders



$\rightarrow \alpha_s^{n+1} \beta_0^n \, n!$

**factorial growth**

→ **big trouble**: the perturbative series is not convergent but only asymptotic

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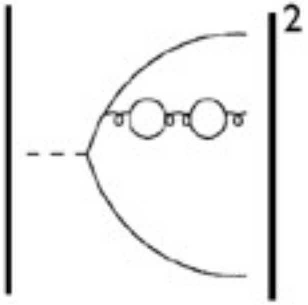

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factorial growth

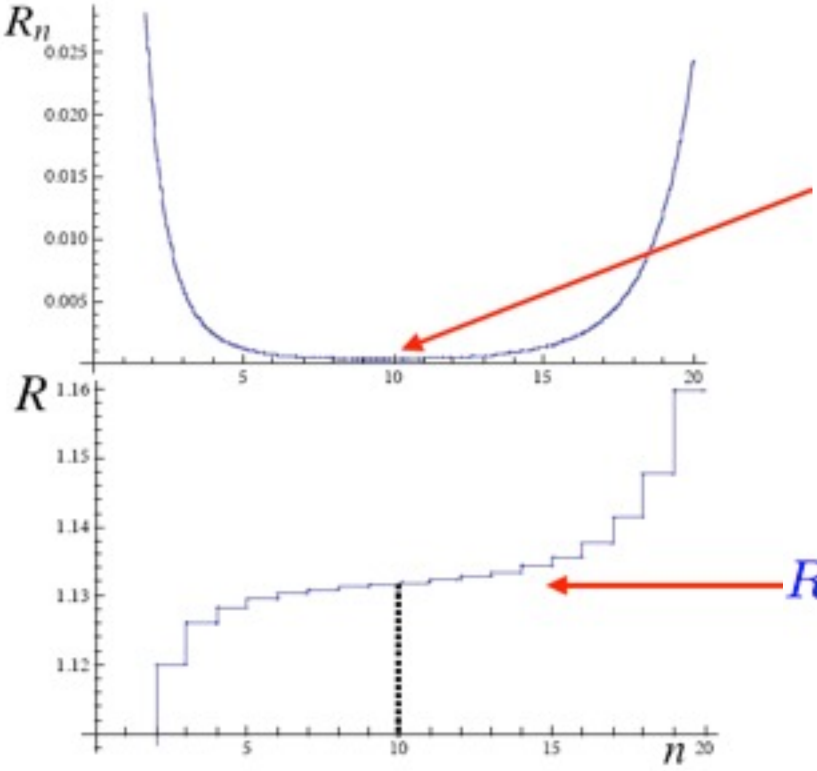
→ **big trouble**: the perturbative series is not convergent but only asymptotic

**illustration:**

try resumming

$$R = \sum_{n=0}^{\infty} \alpha_s^n n!$$

[with $\alpha_s$= 0.1]



**minimal term**

$R_{min} = 1/\alpha_s$

**asymptotic value of the sum:**

$$R_{asymp} = \sum_{n=0}^{n_{min}} \alpha_s^n n!$$

taken from M. Cacciari

# pQCD – non-perturbative bridge

- “renormalon ambiguity” ↔ incompleteness of pQCD series

  → we can only define what the sum of the perturbative series is like truncating it at the minimal term

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- QCD: NP corrections are power suppressed:

$$\boxed{R^{NP} = \exp\left( -p \ln \frac{Q^2}{\Lambda^2}\right) = \left(\frac{\Lambda^2}{Q^2}\right)^p}$$

the value of p depends on the process and can sometimes be predicted

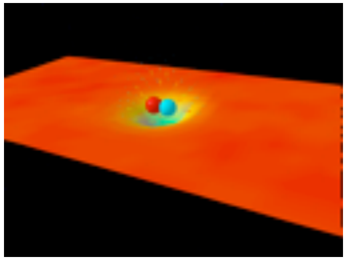
# SUMMARY & OUTLOOK

# QCD: the most perfect gauge theory (so far)

- simple $\mathcal{L}$ but rich & complex phenomenology; few parameters
- in principle complete up to the Planck scale
  (issue: CP, axions?)
- highly non-trivial ground state responsible for all the structure in the visible universe
- **emergent phenomena**: confinement, chiral symmetry breaking, hadrons

- **confinement**

D. Leinweber



non-perturbative structure of hadrons

e.g. through lattice QCD

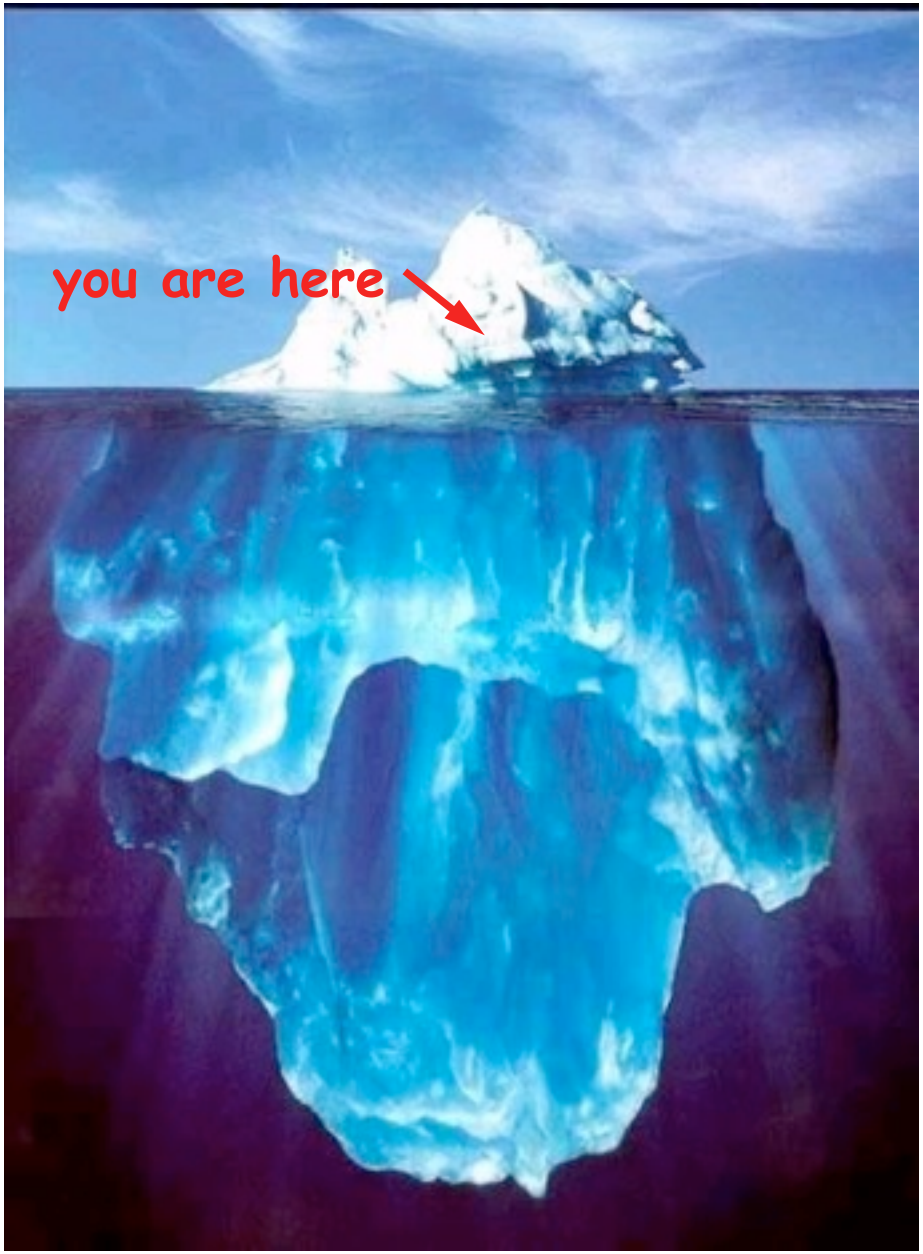
**interplay between High Energy and Hadron Physics**

**asymptotic freedom**

hard scattering cross sections and renormalization group

**perturbative methods**

we have just explored the tip of the iceberg



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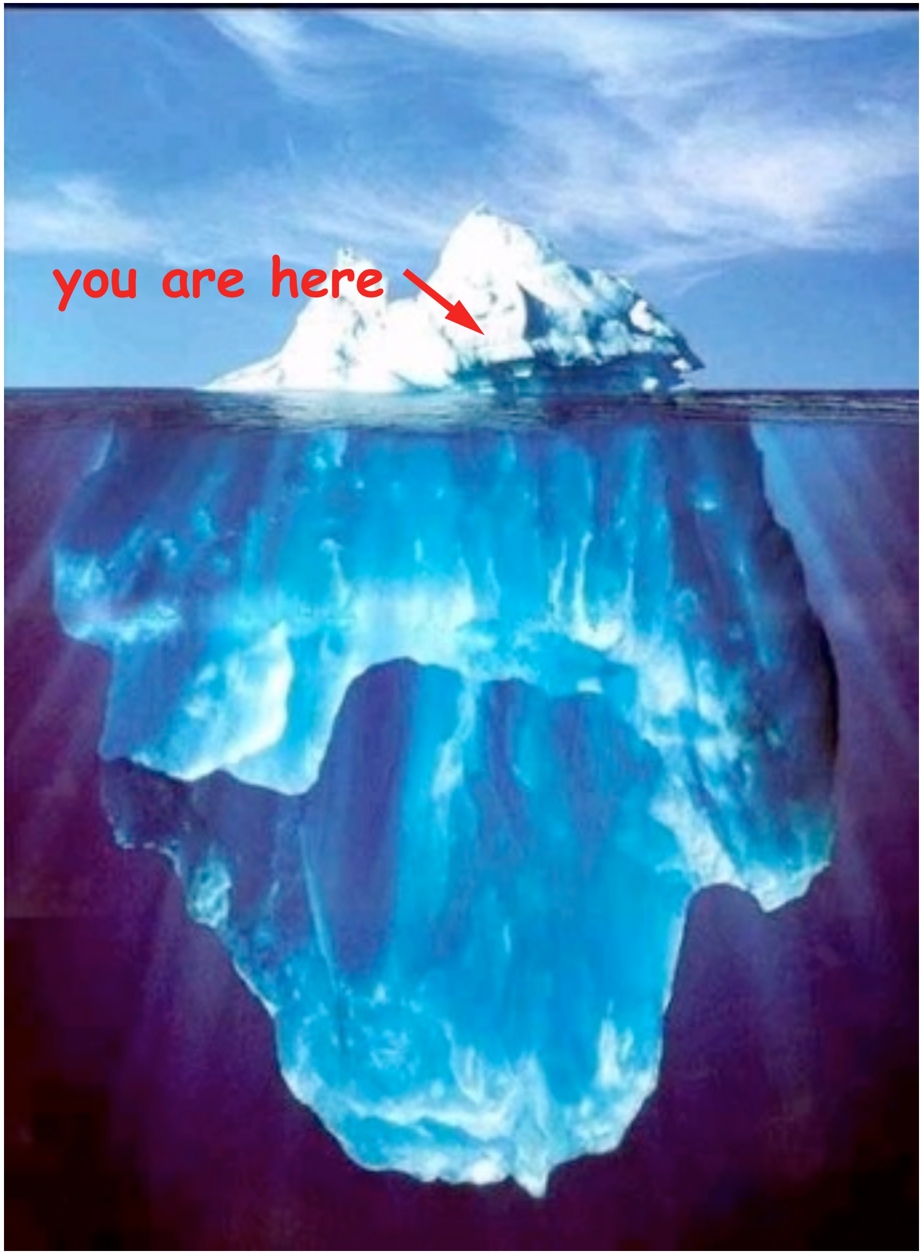

enjoy the other lectures !