Quantum Chromodynamics at High Temperature

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Outline

1. QCD, Deconfinement, Heavy Ion Collisions
2. QCD at finite T, Medium effects, Lattice QCD
3. Out-of-equilibrium systems, Thermalization

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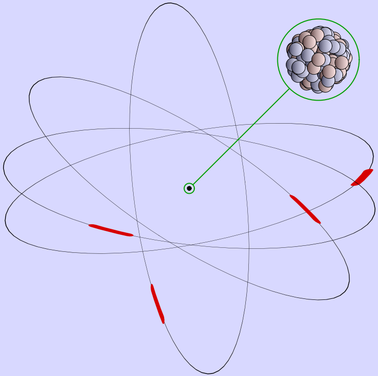
1. **QCD, Deconfinement, Heavy Ion Collisions**
2. QCD at finite T, Medium effects, Lattice QCD
3. Out-of-equilibrium systems, Thermalization

QCD and
Strong Interactions

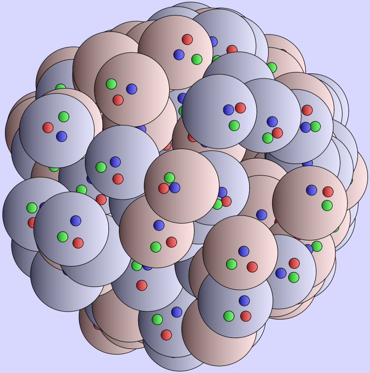
From atoms to nuclei, to quarks and gluons

10^{-10} m : atom (99.98% of the mass is in the nucleus)



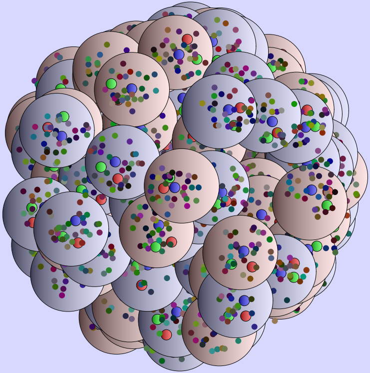
From atoms to nuclei, to quarks and gluons

$< 10^{-15}$ m : quark substructure



From atoms to nuclei, to quarks and gluons

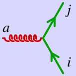
$< 10^{-15}$ m : quarks + gluons



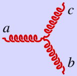
Quarks and gluons

Strong interactions : Quantum Chromo-Dynamics

- Matter : quarks ; Interaction carriers : gluons



$\sim \; g\,(t^a)_{ij}$



$\sim \; g\,(T^a)_{bc}$

- i, j : quark colors ; a, b, c : gluon colors
- $(t^a)_{ij}$: 3×3 SU(3) matrix ; $(T^a)_{bc}$: 8×8 SU(3) matrix

Lagrangian

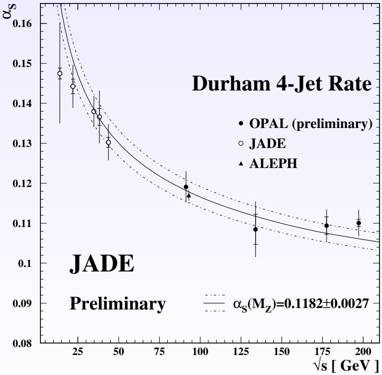
$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_f \overline{\psi}_f(i\not{D} - m_f)\psi_f$$

- Free parameters : quark masses m_f, scale Λ_{QCD}

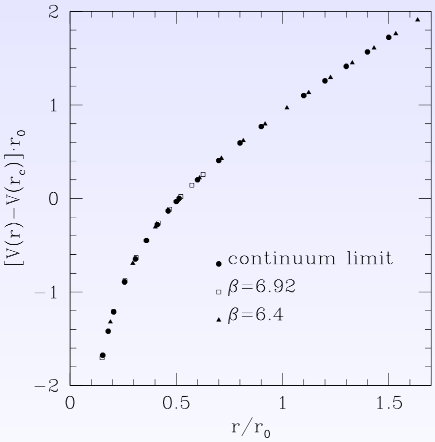
Asymptotic freedom

Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(1/r\Lambda_{QCD})}$$



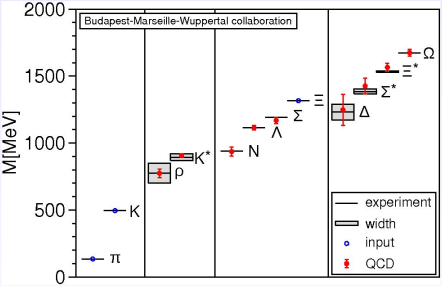
Color confinement



- The quark potential increases linearly with distance

Color confinement

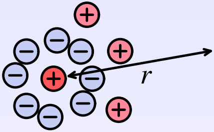
- In nature, we do not see free quarks and gluons (the closest we have to actual quarks and gluons are jets)
- Instead, we see hadrons (quark-gluon bound states):



- The hadron spectrum is uniquely given by $\Lambda_{\rm QCD}, m_f$
- But this dependence is non-perturbative (it can now be obtained fairly accurately by lattice simulations)

Deconfinement

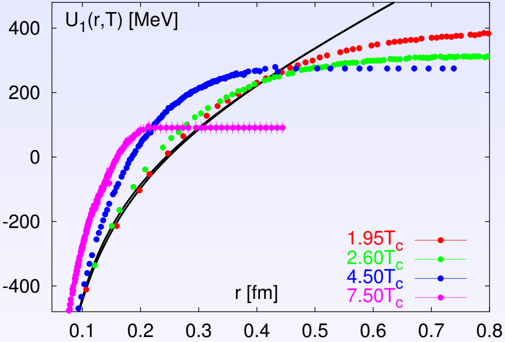
Debye screening at high density



$$V(r) = \frac{\exp(-r/r_{\text{debye}})}{r}$$

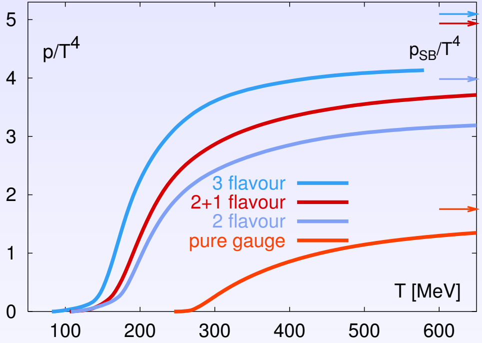
- In a dense environment, color charges are screened by their neighbours
- The Coulomb potential decreases exponentially beyond the Debye radius r_{debye}
- Bound states larger than r_{debye} cannot survive

Debye screening



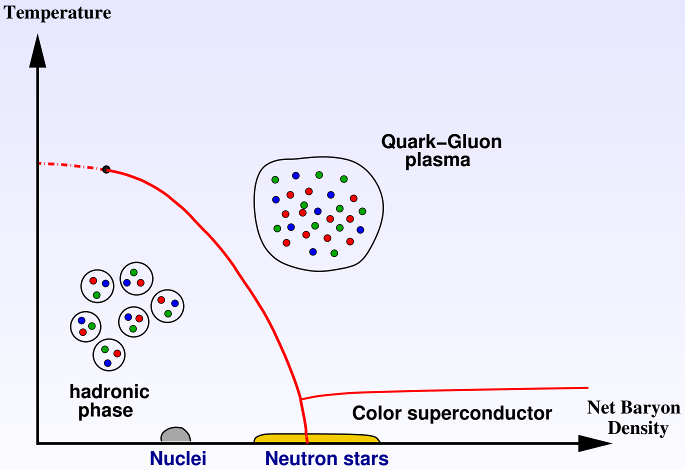
- In lattice calculations, one sees the $q\bar{q}$ potential flatten at long distance as T increases

Deconfinement transition

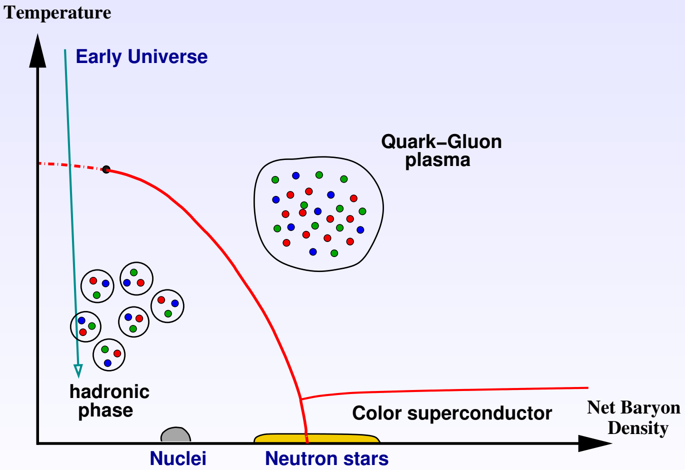


- Fast increase of the pressure :
  - at $T \sim 270$ MeV, if there are only gluons
  - at $T \sim 150$–170 MeV, depending on the number of light quarks

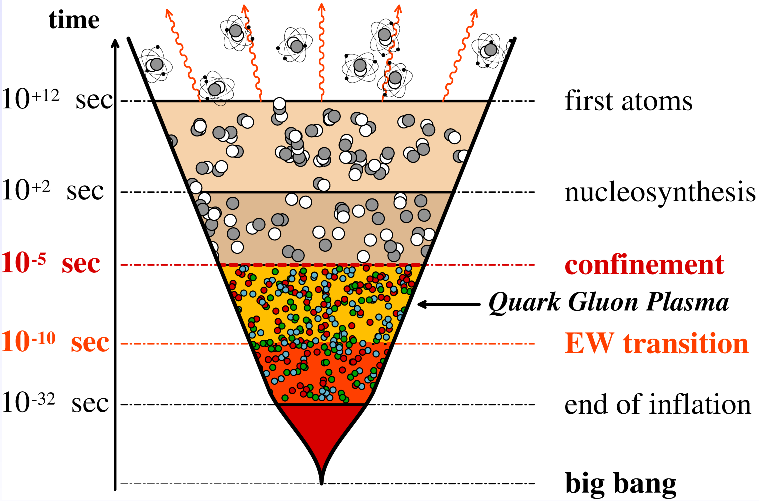
QCD phase diagram



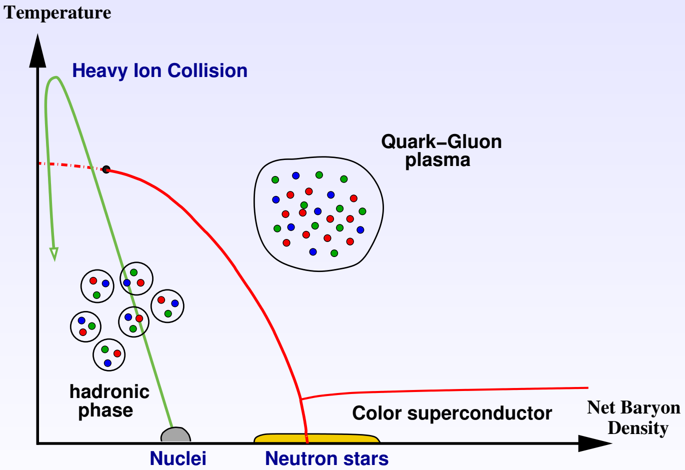
QGP in the early universe



QGP in the early universe



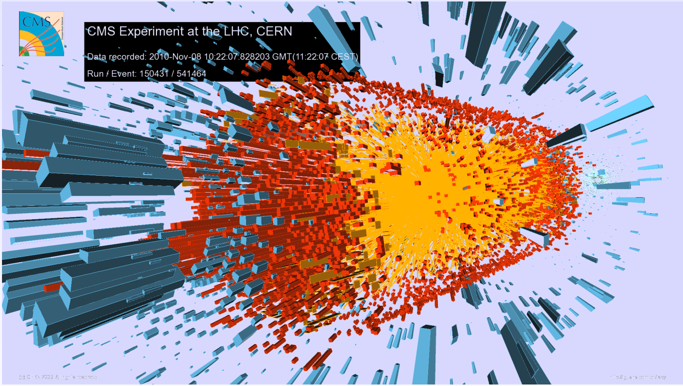
Heavy ion collisions



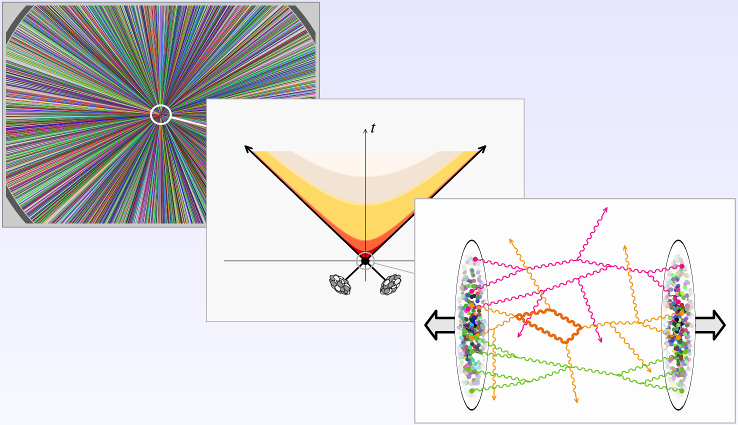
Heavy Ion Collisions

Experimental facilities : RHIC and LHC

Heavy ion collision at the LHC

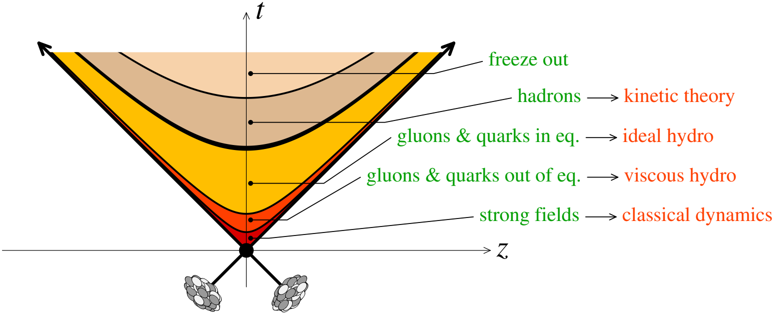


From measured hadrons back to QCD...



Goal : from the final state particles (hadrons), understand the microscopic dynamics of the quarks and gluons

Stages of a nucleus-nucleus collision

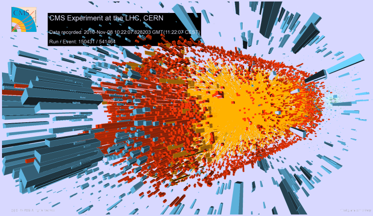


The long way from QCD to the real world...

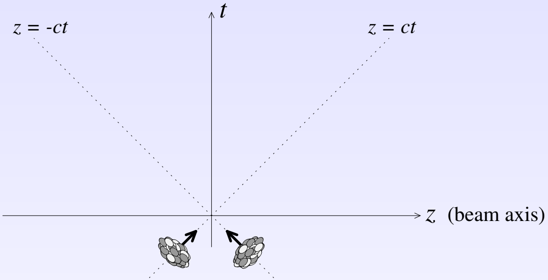
What we can calculate

$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f$$

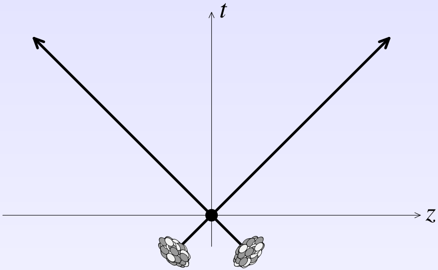
What we can measure



Stages of a nucleus-nucleus collision

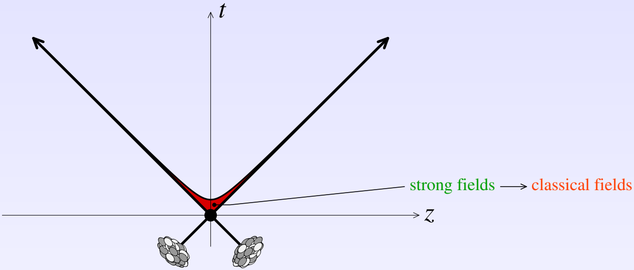


Stages of a nucleus-nucleus collision



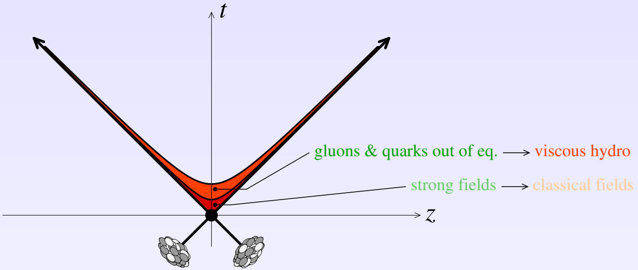
- $\tau \sim 0$ fm/c
- Production of hard particles :
  - jets, direct photons
  - heavy quarks
- calculable with perturbative QCD (leading twist)

Stages of a nucleus-nucleus collision



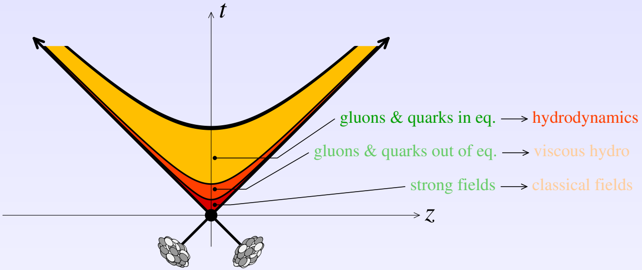
- $\tau \sim 0.2$ fm/c
- Production of semi-hard particles : gluons, light quarks
- relatively small momentum : $p_\perp \lesssim 2\text{–}3$ GeV
- make up for most of the multiplicity
- sensitive to the physics of saturation (higher twist)

Stages of a nucleus-nucleus collision



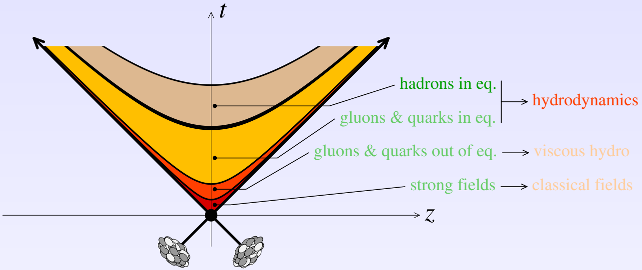
- $\tau \sim$ 1–2 fm/c
- Thermalization
  - some data suggest a fast thermalization
  - but this is still not fully understood from QCD

Stages of a nucleus-nucleus collision



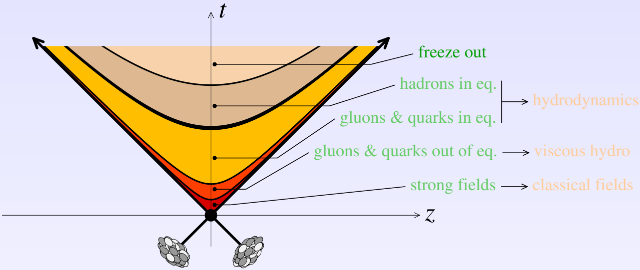
- $2 \leq \tau \lesssim 10$ fm/c
- Quark gluon plasma

Stages of a nucleus-nucleus collision



- $10 \lesssim \tau \lesssim 20$ fm/c
- Hot hadron gas

Stages of a nucleus-nucleus collision



- $\tau \to +\infty$
- Chemical freeze-out :
  density too small to have inelastic interactions
- Kinetic freeze-out :
  no more elastic interactions

Outline

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2. **QCD at finite T, Medium effects, Lattice QCD**
3. Out-of-equilibrium systems, Thermalization

QCD at Finite T

Goals

Examples of questions one would like to answer :

- What are the energy density and the pressure of the quark-gluon plasma at a given temperature?
- What is its viscosity/conductivity/...?
- What photon/dilepton/... spectrum does it radiate?
- If an energetic quark/gluon travels through the QGP, how much energy does it lose?

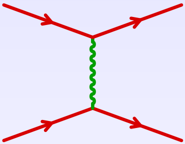
QCD at finite temperature is a set of techniques designed to compute expectations values of the form :

$$\langle \mathcal{O} \rangle \equiv \frac{\mathrm{tr}\, e^{-\beta H}\, \mathcal{O}}{\mathrm{tr}\, e^{-\beta H}}$$

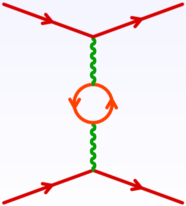
Reminder : $T = 0$

Quantum field theory at T=0

- It can be used to calculate scattering amplitudes, such as $\big\langle \vec{p}_1 \vec{p}_{2\,\mathrm{out}} \big| \vec{k}_1 \vec{k}_{2\,\mathrm{in}} \big\rangle$



- Besides the incoming/outgoing particles, the only other fields that can be involved in the scattering process are quantum fluctuations of the vacuum



Quantum field theory at T=0

- A Quantum Field Theory is specified by its Lagrangian, that describes the interactions among its constituents. E.g.,

$$\mathcal{L} \equiv \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}_{\text{free theory}} + \underbrace{\frac{g^2}{4!}\phi^4}_{\text{interactions}}$$

- When the interactions are weak, one can compute observables in perturbation theory, i.e. as a series in the coupling constant g^2
- LSZ reduction formulas : scattering amplitudes are obtained from the Fourier transform of the time-ordered correlators of the elementary fields. Example :

$$\big\langle \vec{p}_1 \vec{p}_{2\,\mathrm{out}} \big| \vec{k}_1 \vec{k}_{2\,\mathrm{in}} \big\rangle = \int_{x_1, x_2, y_1, y_2} e^{i(k_1\cdot x_1 + k_2\cdot x_2 - p_1\cdot y_1 - p_2\cdot y_2)}$$
$$\times \Box_{x_1} \Box_{x_2} \Box_{y_1} \Box_{y_2} \underbrace{\big\langle 0_{\mathrm{out}} \big| T\, \phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2) \big| 0_{\mathrm{in}} \big\rangle}_{\text{can be calculated perturbatively}}$$

Note : T = time ordering

Quantum field theory at T=0

- The g^2 dependence can be extracted by writing the Heisenberg fields in terms of fields of the interaction representation :

$$\phi(x) \equiv U(-\infty, x^0)\phi_{in}(x)U(x^0, -\infty)$$

$$U(t_2, t_1) = T\ \exp\ i \int_{t_1}^{t_2} d^4x\ \underbrace{\mathcal{L}_I(\phi_{in}(x))}_{\text{interaction term, e.g. } g^2\phi_{in}^4(x)}$$

- One gets a series in g^2 by expanding the exponential in the evolution operator

- Feynman rules in coordinate space :
  - Vertices : $-i\, g^2 \int d^4x$
  - Propagators : $G_F^0(x, y) = \langle 0 | T\, \phi_{in}(x)\phi_{in}(y) | 0 \rangle$

Note : in momentum space,

$$G_F^0(p) \equiv \int d^4(x-y)\ e^{ip\cdot(x-y)}\ G_F^0(x, y) = \frac{i}{p^2 - m^2 + i\epsilon}$$

Quantum field theory at T=0 - Exercises

- Properties of $U(t_1, t_2)$:
  - $U(t,t) = 1$
  - $UU^\dagger = 1$
  - $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$
  - $U^{-1}(t_1, t_2) = U(t_2, t_1)$
- $\phi(x)$ and $\phi_{\rm in}(x)$ coincide when $x^0 \to -\infty$
- If $\phi(x)$ obeys the equation of motion with interactions, then $\phi_{\rm in}(x)$ is a free field :

$$(\square + m^2)\phi(x) - \frac{\partial \mathcal{L}_I(\phi(x))}{\partial \phi(x)} = U(-\infty, x^0)\Big[(\square + m^2)\phi_{\rm in}(x)\Big]U(x^0, -\infty)$$

Perturbative Expansion at $T > 0$

Thermal correlators and their $T = 0$ limit

- Can we generalize the $T = 0$ techniques to the calculation of the following correlators?

$$G(x_1, \cdots, x_n) \equiv \frac{\mathrm{Tr}\left(e^{-\beta H}\, T\, \phi(x_1) \cdots \phi(x_n)\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$

- In terms of eigenstates of the Hamiltonian :

$$G(x_1, \cdots, x_n) = \frac{1}{\mathrm{Tr}\left(e^{-\beta H}\right)} \sum_{\text{states } n} e^{-\beta E_n} \left\langle n \middle| T\, \phi(x_1) \cdots \phi(x_n) \middle| n \right\rangle$$

- When $T \to 0$ (i.e. $\beta \to +\infty$), only the vacuum state $\left|0\right\rangle$ survives since it has the lowest energy. Thus

$$\lim_{T \to 0} G(x_1, \cdots, x_n) = \left\langle 0 \middle| T\, \phi(x_1) \cdots \phi(x_n) \middle| 0 \right\rangle$$

Perturbative expansion

- In order to perform the perturbative expansion at finite T, we must identify all the sources of g dependence
- One of them is the interactions in the evolution of the field operator $\phi(x)$. This is identical to $T = 0$:

$$\phi(x) = U(-\infty, x^0)\phi_{in}(x)U(x^0, -\infty)$$

$$U(t_2, t_1) \equiv T \exp i \int_{t_1}^{t_2} d^4x \, \mathcal{L}_I(\phi_{in}(x))$$

- At $T > 0$, another source of g-dependence is the density operator $\exp(-\beta H)$, since $H = H_0 + H_I$. One can prove

$$e^{-\beta H} = e^{-\beta H_0} \underbrace{T \exp i \int_{-\infty}^{-\infty - i\beta} d^4x \, \mathcal{L}_I(\phi_{in}(x))}_{U(-\infty - i\beta, -\infty)}$$

Perturbative expansion - Exercise

- Proof of
$$\underbrace{\exp(-\beta H)}_{A(\beta)} = \underbrace{\exp(-\beta H_0)\, U(-\infty - i\beta, -\infty)}_{B(\beta)}$$

- $B(\beta)$ can be rewritten as
$$B(\beta) = e^{-\beta H_0}\ T \exp -i \int_{-\infty}^{-\infty - i\beta} dt\ H_{in}^{I}(t)$$
with $\quad H_{in}^{I}(t) = \exp(iH_0(t+\infty))H_{I} \exp(-iH_0(t+\infty))$

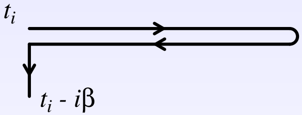
- $A(\beta)$ and $B(\beta)$ are identical at $\beta = 0$ (trivial)
- Their first derivatives are identical at any β
$$A'(\beta) = -H\,A(\beta)$$
$$B'(\beta) = -H_0 B(\beta) - \underbrace{e^{-\beta H_0} H_{in}^{I}(-\infty - i\beta)}_{H_{I}\ e^{-\beta H_0}}\ T \exp -i \int_{-\infty}^{-\infty - i\beta} dt\ H_{in}^{I}(t)$$

Perturbative expansion

- From the previous formulas, we can write :

$$e^{-\beta H}\, T\, \phi(x_1)\cdots\phi(x_n) =$$
$$= e^{-\beta H_0}\, P\, \phi_{in}(x_1)\cdots\phi_{in}(x_n)\ \exp i\int_{\mathcal{C}} d^4x\, \mathcal{L}_I(\phi_{in}(x))$$

$$\mathcal{C} = [t_i, +\infty] \cup [+\infty, t_i] \cup [t_i, t_i - i\beta] \quad :$$



(it is instructive to let the path start at an arbitrary t_i instead of $-\infty$)

- The symbol P denotes path ordering. The contour $\mathcal{C}$ is oriented, and the closest operator to the end of the path should be on the left of the product
- On the upper branch of the contour, the path ordering is equivalent to the usual time-ordering. The times $x_1^0, \cdots, x_n^0$ are on the upper branch of the path

Perturbative expansion

- From the previous formula, one sees that – in coordinate space – perturbation theory at finite T is very similar to perturbation theory at $T = 0$. The only difference is that the time integrations at the vertices run over the contour $\mathcal{C}$

- Feynman rules :

  - Vertices : $-ig \int_{\mathcal{C}} d^4x$

  - Propagator :

$$G^0(x,y) = \frac{\mathrm{Tr}\,(e^{-\beta H_0}\, P\, \phi_{in}(x)\phi_{in}(y))}{\mathrm{Tr}\,(e^{-\beta H_0})}$$

- At the moment, it seems that the result may depend on the arbitrary initial time t_i we have just introduced. However, we will prove shortly that nothing depends on t_i

Perturbative expansion

- The free thermal propagator is obtained from the Fourier decomposition of the free field $\phi_{\rm in}(x)$:

$$\phi_{\rm in}(x)=\int\frac{d^3\vec{\mathbf{p}}}{(2\pi)^3 2E_{\mathbf{p}}}\;\Big[a_{\rm in}(\vec{\mathbf{p}})\,e^{-ip\cdot x}+a^\dagger_{\rm in}(\vec{\mathbf{p}})\,e^{+ip\cdot x}\Big]$$

- Exercise : prove the following relations

$$\begin{aligned}
\big[e^{-\beta H_0},a_{\rm in}(\vec{\mathbf{p}})\big] &= e^{-\beta H_0}(1-e^{-\beta E_{\mathbf{p}}})a_{\rm in}(\vec{\mathbf{p}})\\
{\rm Tr}\,(e^{-\beta H_0}\;a_{\rm in}(\vec{\mathbf{p}})) &= 0\\
{\rm Tr}\,(e^{-\beta H_0}\;a^\dagger_{\rm in}(\vec{\mathbf{p}})a_{\rm in}(\vec{\mathbf{p}}')) &= (2\pi)^3\,2E_{\mathbf{p}}\,n_{_B}(E_{\mathbf{p}})\,\delta(\vec{\mathbf{p}}-\vec{\mathbf{p}}')\\
&\quad \text{with}\;\; n_{_B}(E)=\frac{1}{e^{\beta E}-1}
\end{aligned}$$

- From there, it is easy to obtain :

$$\begin{aligned}
G^0(x,y)=\int\frac{d^3\vec{\mathbf{p}}}{(2\pi)^3 2E_{\mathbf{p}}}\Big[&(\theta_c(x^0-y^0)+n_{_B}(E_{\mathbf{p}}))\,e^{-ip\cdot(x-y)}\\
&+(\theta_c(y^0-x^0)+n_{_B}(E_{\mathbf{p}}))\,e^{+ip\cdot(x-y)}\Big]
\end{aligned}$$

Kubo-Martin-Schwinger symmetry

- The density operator $\exp(-\beta H)$ can be viewed as an evolution operator for an imaginary time shift :

$$e^{-\beta H}\phi(x^0-i\beta,\vec{x})e^{\beta H}=\phi(x^0,\vec{x})$$

- Consider the correlator $\mathcal{G}\equiv \mathrm{Tr}\,(e^{-\beta H}\,T\,\phi(t_i,\vec{x})\cdots)$
- t_i is the "smallest" time on $\mathcal{C}$: $\mathcal{G}=\mathrm{Tr}\,(e^{-\beta H}\,(T\,\cdots)\phi(t_i,\vec{x}))$
- Use the cyclicity of the trace, and the first relation :
$\mathcal{G}=\mathrm{Tr}\,(e^{-\beta H}\,\phi(t_i-i\beta,\vec{x})\,(T\,\cdots))$
- $t_i-i\beta$ is the "largest" time : $\mathcal{G}=\mathrm{Tr}\,(e^{-\beta H}\,T\,\phi(t_i-i\beta,\vec{x})\cdots)$

Kubo-Martin-Schwinger symmetry :

$$\mathcal{G}(\cdots t_i\cdots)=\mathcal{G}(\cdots t_i-i\beta\cdots)$$

Path deformations

- The free propagator does not depend explicitly on t_i.
  It verifies the KMS symmetry
- Any graph contributing to a correlator $\mathcal{G}(x_1, \cdots, x_n)$ has a contribution of the form :

$$\mathcal{G} = \int_{\mathcal{C}} dy_1^0 \cdots dy_p^0 \; F(x_1, \cdots, x_n; y_1^0, \cdots, y_p^0)$$

  - F takes identical values at $y_i^0 = t_i$ and $y_i^0 = t_i - i\beta$
  - F does not depend explicitly on t_i

$\mathcal{G}$ does not depend on t_i

Interpretation : t_i is the time at which the system is put in thermal equilibrium. By definition of thermal equilibrium, no measurement made afterwards can tell the value of t_i

More general path deformations also leave the result unchanged

Conserved charges and chemical potential

- A field ϕ is charged (with charge q) under the operator Q if it obeys a relation of the form $[Q, \phi_{\rm in}(x)] = -q\phi_{\rm in}(x)$

  Note : Q is Hermitian and q is real. If the field ϕ is Hermitian, then q can only be zero. The simplest charged fields are complex scalars :

$$\phi_{\rm in}(x) = \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3 2E_{\mathbf{p}}} \left[a_{\rm in}(\vec{\mathbf{p}})\, e^{-ip\cdot x} + b^\dagger_{\rm in}(\vec{\mathbf{p}})\, e^{+ip\cdot x}\right]$$

- A conserved charge Q has an associated chemical potential μ. Equilibrium expectation values should be calculated with the density operator $\exp(-\beta(H+\mu Q))$

KMS symmetry with conserved charges

$$\mathcal{G}(\cdots t_i \cdots) = e^{\beta\mu q}\, \mathcal{G}(\cdots t_i - i\beta \cdots)$$

Conserved charges

- Exercise : derive the relations :

$$\begin{aligned}
\mathrm{Tr}\,(e^{\beta(H_0+\mu Q)}\,a_{in}^\dagger(\vec{p})a_{in}(\vec{p}\,')) &= (2\pi)^3\,2E_p\,\frac{1}{e^{\beta(E_p-\mu q)}-1}\delta(\vec{p}-\vec{p}\,')\\
\mathrm{Tr}\,(e^{\beta(H_0+\mu Q)}\,b_{in}^\dagger(\vec{p})b_{in}(\vec{p}\,')) &= (2\pi)^3\,2E_p\,\frac{1}{e^{\beta(E_p+\mu q)}-1}\delta(\vec{p}-\vec{p}\,')
\end{aligned}$$

(all the other averages are zero)

- The free propagator now depends on μ :

$$G^0(x,y)=\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\Big[\Big(\theta_c(x^0-y^0)+\frac{1}{e^{\beta(E_p-\mu q)}-1}\Big)\,e^{-ip\cdot(x-y)}$$
$$+\Big(\theta_c(y^0-x^0)+\frac{1}{e^{\beta(E_p+\mu q)}-1}\Big)\,e^{+ip\cdot(x-y)}\Big]$$

Fermions

- Consider a spin 1/2 fermion :

$$\psi_{\rm in}(x) = \sum_{s=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left[b^s_{\rm in}(\vec{p}) u^s(\vec{p})\, e^{-ip\cdot x} + d^{s\dagger}_{\rm in}(\vec{p}) v^s(\vec{p})\, e^{+ip\cdot x} \right]$$

with $(\not{p} - m)u^s(\vec{p}) = 0\,, \quad (\not{p} + m)v^s(\vec{p}) = 0$

- For consistency, fermions must be quantized with anti-commutation relations ▷ Fermi-Dirac distributions

- Free propagator :

$$\begin{aligned} S^0(x,y) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \Big[&(E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)\Big(\theta_c(x^0-y^0) - \frac{1}{e^{\beta(E_p-\mu q)}+1}\Big)\, e^{-ip\cdot(x-y)} \\ &+ (-E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)\Big(\theta_c(y^0-x^0) - \frac{1}{e^{\beta(E_p+\mu q)}+1}\Big)\, e^{+ip\cdot(x-y)} \Big] \end{aligned}$$

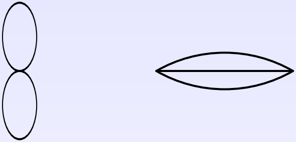
KMS for fermions

$$\mathcal{G}(\cdots t_i \cdots) = -e^{\beta\mu q}\, \mathcal{G}(\cdots t_i - i\beta \cdots)$$

What can we calculate?

Thermodynamical quantities

- Vacuum diagrams are diagrams without any external legs



- The sum of all the vacuum diagrams provides the partition function

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

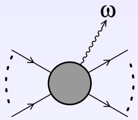
- From Z, one can obtain other thermodynamical quantities :

$$E = -\frac{\partial Z}{\partial \beta}$$
$$S = \beta E + \ln(Z)$$
$$F = E - TS = -\frac{1}{\beta}\ln(Z)$$

Production rates

- A hot plasma of electrically charged particles radiates photons
- Photons do not feel the strong interactions. They escape from the system
- Pedestrian approach :

$$\omega\frac{dN_\gamma}{dt\,dV\,d^3\vec{q}} \propto \int_{\left(\substack{\text{unobserved}\\ \text{particles}}\right)} \left| \quad \right|^2$$



$$\times\, n(\omega_1)\cdots n(\omega_n)$$

$$\times\,(1\pm n(\omega_1'))\cdots(1\pm n(\omega_p'))$$

- Using QFT at finite temperature :

$$\omega\frac{dN_\gamma}{dt\,dV\,d^3\vec{q}} \propto \frac{1}{e^{\omega/T}-1}\;\mathrm{Im}\;\underbrace{\Pi^{\mu}{}_{\mu}(\omega,\vec{q})}_{\text{photon self-energy}}$$

Transport coefficients

- Transport coefficients characterize the ability of the QGP to move certain quantities around :
  - Color conductivity (color charge)
  - Electrical conductivity (electrical charge)
  - Viscosity (momentum)

Green-Kubo formulas :

$$\begin{bmatrix}\text{transport}\\ \text{coefficient}\end{bmatrix} \sim \lim_{\omega\to 0} \frac{1}{\omega}\,\mathrm{Im}\int_0^{+\infty} dt\, d^3\vec{x}\; e^{-i\omega t}\,\left\langle J(t,\vec{x})\,J(0,\vec{0})\right\rangle$$

- J = current that couples to the quantity we want to transport
- $J(0,\vec{0})$ excites the system at $0,\vec{0}$
- $J(t,\vec{x})$ measures the response at $t,\vec{x}$

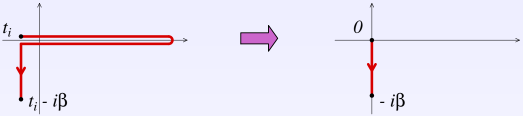
Matsubara Formalism

Thermodynamical quantities

- Vacuum diagrams are pure numbers (they do not depend on any external time)

  ▷ For this reason, we are not tied to using a contour $\mathcal{C}$ that contains the real axis

- We can deform the contour to make it simpler



- If we denote $x^0 = -i\tau$, the variable τ is real and spans the range $[0, \beta]$. The Feynman rules obtained with this choice of the contour $\mathcal{C}$ are known as "Matsubara formalism"

- Note : one could in principle use them to calculate non-vacuum diagrams, but an analytic continuation is necessary to go back to real times (complicated in general)

Matsubara frequencies

- The propagator – and more generally the integrand for any diagram – is β-periodic in the imaginary time τ
- Therefore, one can go to Fourier space by decomposing the time dependence in Fourier series and by doing an ordinary Fourier transform in space :

$$G^0(\tau_x, \vec{x}, \tau_y, \vec{y}) = T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\omega_n(\tau_x-\tau_y)} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\, G^0(\omega_n, \vec{p})$$

with $\omega_n \equiv 2\pi n T$. Note : for fermions, $\omega_n = 2\pi(n+\frac{1}{2})T$
If the line carries the conserved charge q, one must shift $\omega_n \to \omega_n - i\mu q$

Exercise : an explicit calculation gives :

$$G^0(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + \vec{p}^2 + m^2}$$

Matsubara formalism

- Feynman rules :
  - Propagators : $G^0(\omega_n, \vec{\mathbf{p}}) = 1/(\omega_n^2 + \vec{\mathbf{p}}^2 + m^2)$
  - Vertices : g $\quad$ ($\sum \omega_n$ and $\sum \vec{\mathbf{p}}$ conserved at the vertices)
  - Loops : $T \sum_n \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3}$

- Examples (written here in the massless case) :

$$= \lambda T^2 \sum_{m,n} \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3} \frac{d^3\vec{\mathbf{q}}}{(2\pi)^3} \; \frac{1}{(\omega_m^2 + \vec{\mathbf{p}}^2)(\omega_n^2 + \vec{\mathbf{q}}^2)}$$

$$= g^2 T^2 \sum_{m,n} \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3} \frac{d^3\vec{\mathbf{q}}}{(2\pi)^3} \; \frac{1}{(\omega_m^2 + \vec{\mathbf{p}}^2)(\omega_n^2 + \vec{\mathbf{q}}^2)(\omega_{m+n}^2 + (\vec{\mathbf{p}} + \vec{\mathbf{q}})^2)}$$

Calculation of the discrete sums

- The calculation of the discrete sums can be quite hard...

- Method 1 : replace each propagator by

$$G^0(\omega_n, \vec{\mathbf{p}}) = \frac{1}{2E_{\mathbf{p}}} \int_0^\beta d\tau\, e^{-i\omega_n\tau} \Big[(1+n_B(E_{\mathbf{p}}))\, e^{-E_{\mathbf{p}}\tau} + n_B(E_{\mathbf{p}})\, e^{E_{\mathbf{p}}\tau} \Big]$$

- One should combine this trick with the formula

$$\sum_n e^{i\omega_n\tau} = \beta \sum_n \delta(\tau - n\beta)$$

which turns all the time dependence into combinations of delta functions. Then, all the time integrations are trivial

Calculation of the discrete sums

- Method 2 : use a function $P(\omega)$ that has simple poles of residue 1 at each $i\omega_n$. Then, write the discrete sums as

$$\sum_n f(i\omega_n) = \oint_\gamma \frac{dz}{2i\pi}\, f(z)P(z)$$

where γ is a path made of a small circle around each pole

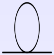
Note : for instance $P(z) = \dfrac{\beta}{e^{\beta z} - 1}$

- If the function $f(z)$ has no pole on the imaginary axis, deform the contour γ in two lines along the imaginary axis

- Deform the contour to bring it along the real energy axis (beware of the poles lying away from the real axis!)

Example

- Exercise. Tadpole in a $\lambda\phi^4$ theory :



$$
\begin{aligned}
&= \frac{\lambda T}{2}\sum_n \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3}\,\frac{1}{\omega_n^2+\vec{\mathbf{p}}^2}\\
&= \frac{\lambda T}{2}\sum_n \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\int_0^\beta d\tau\; e^{-i\omega_n\tau}\Big[(1+n_B(E_{\mathbf{p}}))e^{-E_{\mathbf{p}}\tau}+n_B(E_{\mathbf{p}})e^{E_{\mathbf{p}}\tau}\Big]\\
&= \frac{\lambda}{2}\int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3 2E_{\mathbf{p}}}\int_0^\beta d\tau\sum_n \delta(\tau-n\beta)\Big[(1+n_B(E_{\mathbf{p}}))e^{-E_{\mathbf{p}}\tau}+n_B(E_{\mathbf{p}})e^{E_{\mathbf{p}}\tau}\Big]\\
&= \frac{\lambda}{2}\int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3 2E_{\mathbf{p}}}\big[1+2n_B(E_{\mathbf{p}})\big]
\end{aligned}
$$

(the remaining integral is "elementary")

- Note : in the last formula, the 1 gives the usual ultraviolet divergence, and the n_B gives a finite contribution that vanishes if $T\to 0$ $\triangleright$ this term is a medium effect

Schwinger-Keldysh Formalism

Why we may need something different

- The Matsubara formalism is ideal for thermo-dynamical quantities
- For quantities that depend on energy, one would need to perform an analytic continuation from (discrete) imaginary frequencies to (continuous) real energy
- For 2-point functions, how to do this is well known, but already tricky (how to put the $i\epsilon$'s...?)
- For 3-point functions and beyond, this is usually too complicated, and it is preferable to use a formalism that gives directly the answer in terms of real energies

Schwinger-Keldysh formalism

- Forget for the time being the vertical appendix to the time contour.
  Loose justification : take $t_i \to -\infty$ and unplug the interactions in this limit, then the initial density operator is made of the free Hamiltonian, and there is no need for this extra bit of contour

- Break the propagator $G(x, y)$ in four components, depending on where the times x^0, y^0 are on the contour

- Fourier transform the propagator

$$
\begin{aligned}
G_{++}(p) &= \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi\, f(E_{\boldsymbol{p}})\, \delta(p^2 - m^2) \\
G_{+-}(p) &= 2\pi \left(\theta(-p^0) + f(E_{\boldsymbol{p}})\right) \delta(p^2 - m^2) \\
G_{--}(p) &= [G_{++}(p)]^* \quad , \quad G_{-+}(p) = G_{+-}(-p) \\
&\quad \text{with } f(\omega) \equiv \frac{1}{e^{\beta\omega} - 1} \quad , \quad E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}
\end{aligned}
$$

Schwinger-Keldysh formalism

- Feynman rules :
  - Each vertex can be of type $+$ or $-$
  - Type $+$: $-ig^2$ Type $-$: $+ig^2$
  - Connect a vertices of types ϵ and ϵ' by the propagator $G_{\epsilon\epsilon'}$

- Note : the $T \to 0$ limit of this formalism is equivalent to Cutkosky's cutting rules. In the calculation of the cross-section of inclusive processes, they provide a way to perform the sum over the unobserved particles in the final state

- Ignoring the vertical branch of the contour was a (small) cheat. It leads to slightly incorrect Feynman rules. They can be fixed simply by changing

$$f(E_{\mathbf{p}}) \quad \to \quad f(|p^0|)$$

Exercise

- Check the following formula :

$$\begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix} = \mathcal{U} \begin{pmatrix} G_F & 0 \\ 0 & G_F^* \end{pmatrix} \mathcal{U}$$

with

$$\mathcal{U}(p) \equiv \begin{pmatrix} \sqrt{1+f(E_{\boldsymbol{p}})} & \frac{\theta(-p^0)+f(E_{\boldsymbol{p}})}{\sqrt{1+f(E_{\boldsymbol{p}})}} \\ \frac{\theta(+p^0)+f(E_{\boldsymbol{p}})}{\sqrt{1+f(E_{\boldsymbol{p}})}} & \sqrt{1+f(E_{\boldsymbol{p}})} \end{pmatrix}$$

and

$$G_F(p) \equiv \frac{i}{p^2 - m^2 + i\epsilon}$$

KMS symmetry

- The n-point correlators in the Schwinger-Keldysh formalism obey the following relation :

$$\sum_{\epsilon_1\cdots\epsilon_n=\pm} \Gamma_{\epsilon_1\cdots\epsilon_n}(k_1,\cdots,k_n)=0$$

Note : this relation is true even out of equilibrium

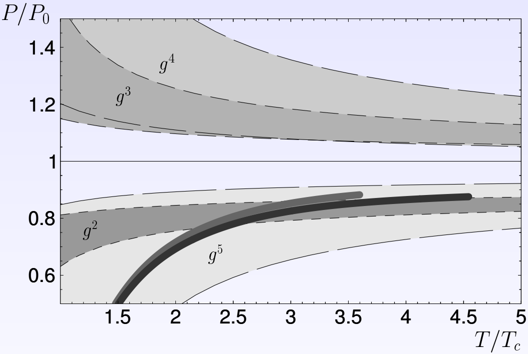
- A second relation - related to KMS - is satisfied in equilibrium :

$$\sum_{\epsilon_1\cdots\epsilon_n=\pm}\Big[\prod_{\{i|\epsilon_i=-\}} e^{-\beta k_i^0}\Big]\,\Gamma_{\epsilon_1\cdots\epsilon_n}(k_1,\cdots,k_n)=0$$

Collective Phenomena in the QGP

Bad convergence of the perturbative expansion

- Example: perturbative calculation of the QGP pressure :



- Does not converge at all...
- The bare quanta of the naive perturbative expansion are quite different from the actual (dressed) quanta in the QGP

Trivial example of what may go wrong

- Consider the free theory of a massive field

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

- Suppose that we are (very) naive and decide to treat the mass term as an interaction... The bare propagator is therefore massless

$$G^0(p) = \frac{i}{p^2}$$

- The perturbative correction of order n in m^2 reads

$$G^n(p) = \frac{i}{p^2} \left[\frac{m^2}{p^2} \right]^n$$

- This is a geometrical series whose sum is $i/(p^2 - m^2)$
- BUT: the domain of convergence of the series is $p^2 > m^2$

Length Scales

Degrees of freedom

- Quarks : $2\,(\mathrm{spin}) \times 3\,(\mathrm{color}) = 6$ (per flavor)

$$\frac{dN_q}{d^3\vec{x}\,d^3\vec{k}} = \frac{1}{e^{\omega/T}+1} \qquad \text{(Fermi-Dirac)}$$

- Gluons : $2\,(\mathrm{spin}) \times 8\,(\mathrm{color}) = 16$

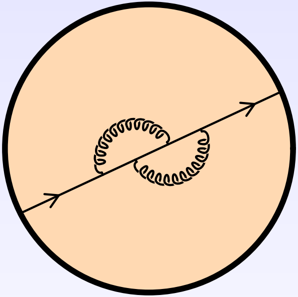
$$\frac{dN_g}{d^3\vec{x}\,d^3\vec{k}} = \frac{1}{e^{\omega/T}-1} \qquad \text{(Bose-Einstein)}$$

- Average energy per particle : $\langle E \rangle \sim T$
- Particle density : $\rho \sim T^3$
- Average distance between particles : $\ell \sim 1/T$

Length scales

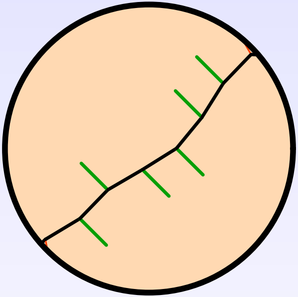
- $1/T$: wavelength of particles in the plasma
- $1/gT$: typical distance for collective phenomena
  - Thermal masses of quasi-particles
  - Screening phenomena
  - Damping of plasma waves
- $1/g^2T$: distance between two small angle scatterings
  - Color transport
  - Photon emission
- $1/g^4T$: distance between two large angle scatterings
  - Momentum, electric charge transport
    $\triangleright$ characteristic scale of hydrodynamic modes
- In the weak coupling limit ($g \ll 1$), there is a clear hierarchy between these scales
- Distinct effective theories according to the characteristic scale of the problem under study

Vacuum fluctuations



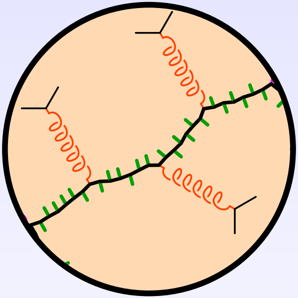
- At distances scales $\ell \lesssim 1/T$, medium effects are irrelevant
- At such scales the dynamics is determined only by QCD vacuum fluctuations

Thermal fluctuations



- Distance scales $1/T \lesssim \ell \lesssim 1/gT$ control the bulk thermodynamic properties. The system can be studied by QCD at finite temperature
- The leading thermal effects can be treated by an effective theory that encompasses the main collective effects, and that has the form of a collision-less Vlasov equation

Small angle scatterings

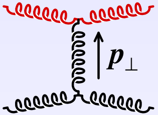


- When it is necessary to follow a plasma particle over distances $1/g^2T \lesssim \ell$, we must take into account soft (small angle) collisions with other particles of the plasma
- This can be done simply by adding a collision term to the previous Vlasov equation

Scattering rate

- Collisional width (up to logs) :

$$\Gamma_{\rm coll} = \Bigg|$$

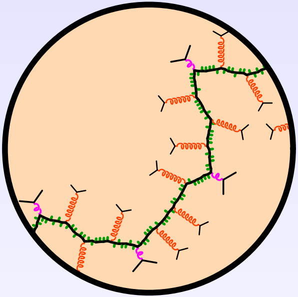


$$\Bigg|^2 \sim g^4\, T^3 \int_{m_{\rm debye}} \frac{d^2\vec{p}_\perp}{p_\perp^4} \sim g^2 T$$

- $\lambda \equiv 1/\Gamma_{\rm coll}$ is the mean free path between two small angle scatterings ($\theta \sim g$)

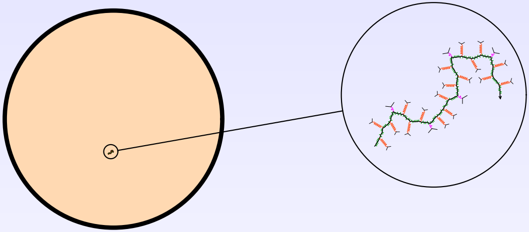
- Note : the mean free path between two large angle scatterings ($\theta \sim 1$) is $\sim 1/g^4 T$

Large angle scatterings



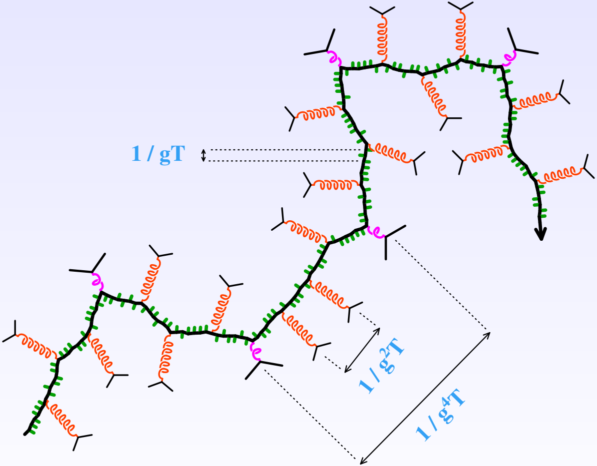
- Over distance scales $\ell \sim 1/g^4T$, one must take into account the large angle collisions, that change significantly the direction of motion of the particle (this is necessary e.g. for calculating transport coefficients)
- The most efficient way to describe the system at these scales is via a Boltzmann equation for color/spin averaged particle distributions

Hydrodynamical regime



- The hydrodynamical regime is reached for length scales that are much larger than the mean free path : $1/g^4T \ll \ell$
- In order to describe the system at such scales, one needs :
  - Hydrodynamical equations (Euler, Navier-Stokes)
  - Conservation equations for the various currents
  - Equation of state, viscosity

Summary



Effective Descriptions

Perturbative modes

- A mode is perturbative if its kinetic energy is much larger than its potential energy
  - Kinetic energy : $\langle K \rangle \sim \langle (\partial A)^2 \rangle \sim k^2 \langle A^2 \rangle$
  - Potential energy : $\langle U \rangle \sim g^2 \langle A^4 \rangle \sim g^2 \langle A^2 \rangle^2$

  ▷ Thus, a mode k is perturbative if $g^2 \langle A^2 \rangle \ll k^2$

- When discussing the order of magnitude of $\langle A^2 \rangle$, it is useful to distinguish the contribution of the various momentum scales by defining

$$\langle A^2 \rangle_{\kappa^*} \sim \int^{\kappa^*} \frac{d^3\vec{p}}{E_p}\, f(E_p)$$

Perturbative modes

- Hard modes : $k \sim T$, $\langle A^2 \rangle_{T} \sim T^2$. Thus, $\langle K \rangle \gg \langle U \rangle$
- Soft modes : $k \sim gT$, $k^2 \sim g^2 \langle A^2 \rangle_{T}$

  But the contribution of soft modes to $\langle A^2 \rangle$ is $\langle A^2 \rangle_{gT} \sim gT^2$, and $k^2 \gg g^2 \langle A^2 \rangle_{gT}$

  The soft modes interact strongly with the hard modes, but weakly among themselves $\triangleright$ they can be described perturbatively after the hard modes have been resummed
- Ultrasoft modes : $k \sim g^2 T$, $\langle A^2 \rangle_{g^2 T} \sim g^2 T^2$, $k^2 \sim g^2 \langle A^2 \rangle_{g^2 T}$

  The dynamics of the ultrasoft modes is completely non-perturbative, because their self-interactions are as large as their kinetic energy

Effective theory for the soft modes

Braaten, Pisarski (1990), Frenkel, Taylor (1990)

- Obtained from the bare perturbative expansion by the resummation of Hard Thermal Loops (HTL) :

$$\Delta\mathcal{L}_{\rm HTL}(\text{gluons}) = \frac{m_g^2}{2}\int \frac{d\Omega_{\hat{v}}}{4\pi}\, F_{\mu\alpha}\frac{v^\alpha v^\beta}{(v\cdot D)^2}F_\beta{}^\mu\ , \qquad v^\mu = (1,\hat{v})$$

- Can be formulated as a (local) collisionless transport theory for classical particles (Blaizot, Iancu (1993-1995)) :

$$(1) \qquad [D_\mu, F^{\mu\nu}] = m_g^2\int\frac{d\Omega_{\hat{v}}}{4\pi}v^\nu W(x,\hat{v})$$

$$(2) \qquad [v\cdot D, W(x,\hat{v})] = \hat{v}\cdot E(x)$$

  - $W(x,\hat{v})$ is the density of hard particles $(\omega\sim T)$ at the location x, with a velocity in the direction $\hat{v}$
  - (1) : Yang-Mills equation for the soft field modes $(\omega\sim gT)$
  - (2) : Vlasov equation for the hard particles

Dimensional reduction

- By summing the Matsubara modes whose frequency is non-zero (fermions, bosons for $n \neq 0$), one gets a 3-dimensional Yang-Mills theory coupled to an adjoint Higgs :

$$\mathcal{L}_E = \frac{1}{4} F_{ij}^2 + \mathrm{tr}[D_i, A_0]^2 + m_E^2 \, \mathrm{tr} A_0{}^2 + \frac{\lambda_E}{2} (\mathrm{tr} A_0{}^2)^2 + \cdots$$

  - A_0 is the gluon zero mode
  - m_E, λ_E are determined by matching to the underlying theory (i.e. QCD)

- By integrating out the massive A_0, one gets a 3-dimensional pure Yang-Mills theory :

$$\mathcal{L}_M = \frac{1}{4} F_{ij}^2 + \cdots$$

  - its coupling g_M is determined order by order from $\mathcal{L}_E$
  - this Yang-Mills theory is non-perturbative, and must be simulated on a lattice (this is much simpler than simulations of 4-dim QCD)

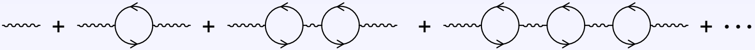
Medium Effects

Dressed propagator

- In order to assess how the medium affects the propagation of gauge excitations, one should compute the polarization tensor

$$\Pi^{\mu\nu}(x,y) \equiv \left\langle J^\mu(x) J^\nu(y) \right\rangle$$

- The leading effect of the medium arises via the 1-loop self-energy. Diagrammatically, this amounts to summing :



- The properties of the medium can be read off the analytic properties of this resummed propagator (cuts, poles, ...)

Dressed propagator

- Reminder : the polarization tensor $\Pi^{\mu\nu}$ is transverse. At $T = 0$, this implies :

$$\Pi^{\mu\nu}(P) = \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right) \Pi(P^2)$$

  - This is due to gauge invariance and Lorentz invariance
  - Exercise : this property ensures that the photon remains massless at all orders of perturbation theory

- This formula is not valid at $T > 0$, because there is a preferred frame (in which the plasma velocity is zero)

  $\triangleright$ the tensorial decomposition of $\Pi^{\mu\nu}$ is more complicated, and the photon may acquire an effective mass

Dressed propagator

- At finite T, the tensorial decomposition of $\Pi^{\mu\nu}$ is :

$$\Pi^{\mu\nu}(P) = P_T^{\mu\nu}(P)\,\Pi_T(P) + P_L^{\mu\nu}(P)\,\Pi_L(P)$$

with the following projectors (in the plasma rest frame)

$$P_T^{ij}(P) = g^{ij} + \frac{p^i p^j}{\vec{p}^2}\,, \quad P_T^{0i}(P) = 0\,, \quad P_T^{00}(P) = 0$$

$$P_L^{ij}(P) = -\frac{(p^0)^2 p^i p^j}{\vec{p}^2 P^2}\,, \quad P_L^{0i}(P) = -\frac{p^0 p^i}{P^2}\,, \quad P_L^{00}(P) = -\frac{\vec{p}^2}{P^2}$$

- This leads to the following resummed propagator :

$$D^{\mu\nu}(P) = P_T^{\mu\nu}(P)\,\frac{1}{P^2 - \Pi_T(P)} + P_L^{\mu\nu}(P)\,\frac{1}{P^2 - \Pi_L(P)}$$

Dressed propagator - Exercise

- Check the following properties of the tensors $P^{\mu\nu}_{T,L}$:

$$P_{T\ \mu}^{\mu} = 2$$

$$P_{L\ \mu}^{\mu} = 1$$

$$P_{T\ \alpha}^{\mu}\, P_{T}^{\alpha\nu} = P_{T}^{\mu\nu}$$

$$P_{L\ \alpha}^{\mu}\, P_{L}^{\alpha\nu} = P_{L}^{\mu\nu}$$

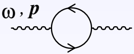
$$P_{T\ \alpha}^{\mu}\, P_{L}^{\alpha\nu} = 0$$

Dressed propagator

- The calculation of $\Pi^{\mu\nu}$ can be done in the Matsubara formalism (i.e. for a discrete imaginary frequency $i\omega_n$ + an analytic continuation $i\omega_p \to p_0$), or directly for real energies in the Schwinger-Keldysh formalism

- Because one is after the long distance properties of the plasma, one also makes the approximation $|\vec{p}| \ll |\vec{k}|$
(Hard Thermal Loops : Braaten, Pisarski - 1990)

- For instance, the fermionic contribution to the spatial part Π^{ij} of the polarization tensor reads :



$$= -\frac{g^2 N_f T}{2} \int \frac{d^3\vec{k}}{(2\pi)^3}\, \widehat{v}_k^i \, \frac{\partial n_F(\vec{k})}{\partial k^l} \left[\delta^{jl} - \frac{\widehat{v}_k^j \widehat{v}_k^l}{\omega - \widehat{v}_k \cdot \vec{p} + i\epsilon} \right]$$

($\widehat{v}_k \equiv \vec{k}/|\vec{k}|$)

  - Note : with the gluon loop, the only change is $N_f \to N_f + 2N_c$

Quasi-particles

- The functions $\Pi_{T,L}(P)$ read (here, for QED) :

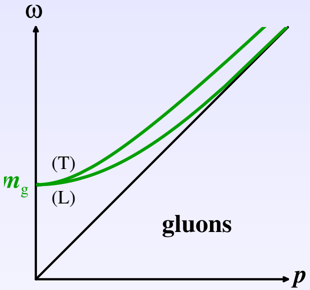
$$\Pi_T(P) = \frac{e^2T^2}{6}\left[\frac{p_0^2}{p^2} + \frac{p_0}{2p}\left(1 - \frac{p_0^2}{p^2}\right)\ln\left(\frac{p_0+p}{p_0-p}\right)\right]$$
$$\Pi_L(P) = \frac{e^2T^2}{3}\left[1 - \frac{p_0^2}{p^2}\right]\left[1 - \frac{p_0}{2p}\ln\left(\frac{p_0+p}{p_0-p}\right)\right]$$

- Quasi-particles correspond to poles in the propagator. Their dispersion relation is the function $p_0 = \omega(\vec{\mathbf{p}})$ that defines the location of the pole

- The inverse of the imaginary part of p_0 is the lifetime of the quasi-particles (If $\mathrm{Im}(p_0) = 0$, they are stable). In order to have well defined quasi-particles, one must have $\mathrm{Im}(p_0) \ll \mathrm{Re}(p_0)$

Quasi-particles

- Dispersion relation of gluons in the plasma :



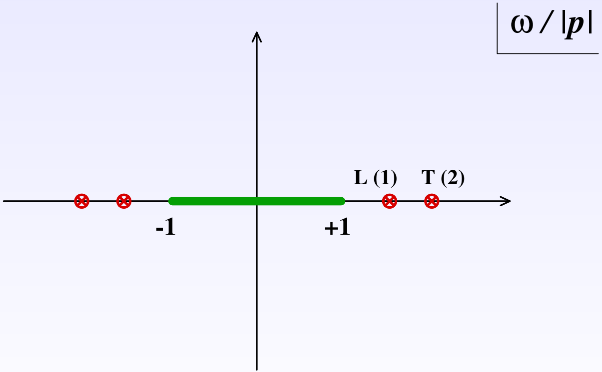
- Thermal masses due to interactions with the other particles in the plasma :

$$m_q \sim m_g \sim gT$$

- At this order, the quasi-particles are stable ($\mathrm{Im}\,\Pi_{T,L} = 0$)

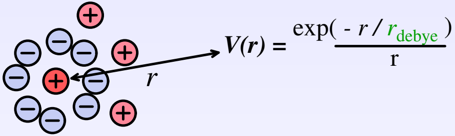
Singularities

- In the complex plane of $\omega/|\vec{\mathbf{p}}|$, the dressed propagator has poles (quasi-particles) and a cut (Landau damping) :



Debye screening

- A test charge polarizes the particles of the plasma in its vicinity, in order to screen its charge :



- The Coulomb potential of the test charge decreases exponentially at large distance. The effective interaction range is :

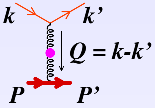
$$\ell \sim 1/m_{debye} \sim 1/gT$$

- Note : static magnetic fields are not screened by this mechanism (they are screened over length-scales $\ell_{mag} \sim 1/g^2T$)

Debye screening

- Place a quark of mass M at rest in the plasma, at $\vec{r}=0$
- Scatter another quark off it. The scattering amplitude reads

$$\mathcal{M}=\left[g\overline{u}(\vec{k}')\gamma_\mu u(\vec{k})\right]\left[g\overline{u}(\vec{P}')\gamma_\nu u(\vec{P})\right]\sum_{\alpha=T,L}\frac{P_\alpha^{\mu\nu}(Q)}{Q^2-\Pi_\alpha(Q)}$$



  - If $\vec{P}=0$ (test charge at rest), only $\alpha=L$ contributes
  - From $(P+Q)^2=M^2$, we get a $2\pi\delta(q_0)/2M$

- For the scattering off an external potential A^μ, the amplitude is $\mathcal{M}=\left[g\overline{u}(\vec{k}')\gamma_\mu u(\vec{k})\right]\,A^\mu(Q)$

- Thus, the potential created by the test charge at rest is :

$$A^\mu(Q)=g\frac{\overline{u}(\vec{P}')\gamma_\nu u(\vec{P})}{2M}\;\frac{2\pi\delta(q_0)P_L^{\mu\nu}(0,\vec{q})}{\vec{q}^{\,2}+\Pi_L(0,\vec{q})}=\frac{2\pi g\delta^{\mu 0}\delta(q_0)}{\vec{q}^{\,2}+\Pi_L(0,\vec{q})}$$

Debye screening

- By a Fourier transform, we obtain the Coulomb potential :

$$A^0(\vec{\mathbf{r}}) = g \int \frac{d^3\vec{\mathbf{q}}}{(2\pi)^3} \; \frac{e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}}{\vec{\mathbf{q}}^2 + \Pi_L(0,\vec{\mathbf{q}})}$$

- If we are in the vacuum, $\Pi_L = 0$, and the Fourier transform gives the usual Coulomb law :

$$A^0_{vac}(\vec{\mathbf{r}}) = g \int \frac{d^3\vec{\mathbf{q}}}{(2\pi)^3} \; \frac{e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}}{\vec{\mathbf{q}}^2} = \frac{g}{4\pi|\vec{\mathbf{r}}|}$$

- In a plasma, $\Pi_L(0,\vec{\mathbf{q}}) = \frac{g^2T^2}{3} \equiv m_D^2$. The Fourier transform can also be done exactly

$$A^0(\vec{\mathbf{r}}) = g \int \frac{d^3\vec{\mathbf{q}}}{(2\pi)^3} \; \frac{e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}}{\vec{\mathbf{q}}^2 + m_D^2} = \frac{g}{4\pi|\vec{\mathbf{r}}|} \; e^{-m_D|\vec{\mathbf{r}}|}$$

$\triangleright$ the potential is unmodified at $r \ll 1/m_D$, but exponentially suppressed at large distance

Debye screening

- It is easy to see here why the naive perturbation theory works pretty badly

- Suppose we want to calculate the Coulomb potential of a test charge in the QGP in perturbation theory. The term of order g^{2n+1} would be :

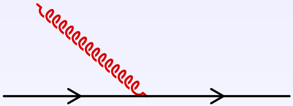
$$A^0_{2n+1}(t,\vec{r}) = (-1)^n \, g \, m_{_D}^{2n} \int \frac{d^3\vec{\mathbf{q}}}{(2\pi)^3} \, \frac{e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}}{\mathbf{q}^{2n+2}}$$

▷ all these corrections are very divergent in the infrared. No truncation in the series over n gives the correct long distance behavior of the potential

Landau damping

- The self-energies $\Pi_{L,T}(p_0, \vec{p})$ have an imaginary part when $|p_0| \le |\vec{p}|$. This implies that the propagation of space-like modes is attenuated

- A wave propagating through the plasma is damped because its quanta may be absorbed by particles of the plasma :



- The characteristic frequency of this damping is :

$$\omega_c \sim gT$$

Lattice QCD

Ref : Z. Fodor and C. Hoelbling, arXiv:1203.4789

Partition function

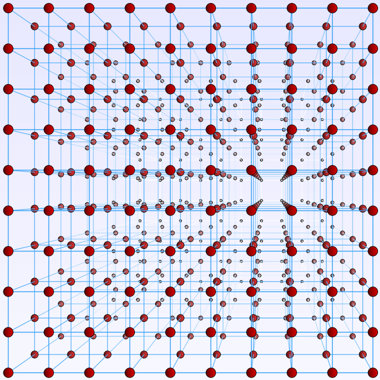
- Partition function :

$$Z \equiv \mathrm{Tr}\,(e^{-\beta H}) = \int [\mathcal{D}A^\mu \mathcal{D}\overline{\psi}\mathcal{D}\psi]\; e^{-S_E[A^\mu,\overline{\psi},\psi]}$$

- The perturbative expansion has a slow convergence
- In the region of the phase transition, the coupling is not small
- Could Z be computed non-perturbatively from first principles?

Lattice QCD

- Discretize the Euclidean space-time on a 4-dim cubic lattice :



- The functional integration becomes an ordinary integral, over a high (but finite) dimensional domain. This can be evaluated by Monte-Carlo sampling, provided the weight $\exp(-S_E)$ is positive

Lattice QCD : gluons

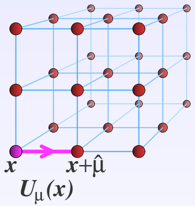
- Naively, one may think of putting the gauge potential A^μ on the nodes of the lattice. Problem : this breaks the gauge invariance of the action by terms proportional to the lattice spacing

- Wilson formulation : introduce a link variable

$$U_\mu(x) \equiv P \exp i g_0 \int_x^{x+\widehat{\mu}} ds\, A^\mu(s)$$

that lives on the edge between the nodes x and $x + \widehat{\mu}$. Under a gauge transformation, it transforms as

$$U_\mu(x) \quad \to \quad \Omega(x)\, U_\mu(x)\, \Omega^\dagger(x+\widehat{\mu})$$



- Wilson action for the gluons :

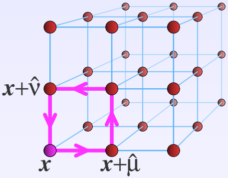
$$S_E = \frac{6}{g_0^2} \sum_{x;\mu\nu} 1 - \frac{1}{3} \mathrm{Re}\,\mathrm{Tr}\,\underbrace{\left(U_\mu(x) U_\nu(x+\widehat{\mu}) U_\mu^\dagger(x+\widehat{\nu}) U_\nu^\dagger(x)\right)}_{\text{plaquette at the point } x \text{ in the } \mu\nu \text{ plane}}$$

Lattice QCD : gluons

Properties of the Wilson action :

- Gauge invariant
- Goes to the continuum action when $a \to 0$

  Note : there exist "improved" actions for which this convergence is faster



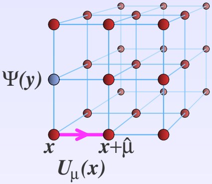
Lattice QCD : fermions

Spinors live on the nodes of the lattice
Under gauge transformations: $\psi(x) \to \Omega(x)\psi(x)$
Covariant derivative :

$$D_\mu \psi(x) = U_\mu(x)\psi(x+\widehat{\mu}) - \psi(x)$$

Gauge transformation :

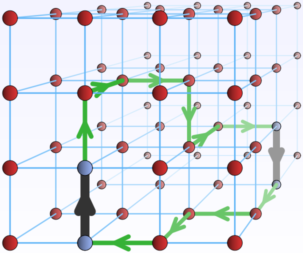
$$D_\mu \psi(x) \quad \to \quad \Omega(x) D_\mu \psi(x)$$



- The action is quadratic in the spinors $\Longrightarrow$ integrate them out
  - Contractions between the ψ's and $\psi^\dagger$'s from the correlator (e.g. a current-current correlator) one evaluates lead to Dirac propagators $(\not{D} + m)^{-1}$. On the lattice, large matrix inversion
  - This also gives the Dirac determinant $\mathrm{Det}\,(\not{D} + m)$. On the lattice, this is computed by stochastic methods

Lattice QCD : quenched approximation

- The computation of the determinant is very expensive, because it must be taken into account in the update of the gauge configurations
- Quenched approximation : ignore the determinant. This amounts to assuming that the quarks running in the loops are very heavy
- The only fermions left are the Dirac propagators connecting the fermions in the operators one evaluates. E.g., for a meson-meson correlator, one would have:

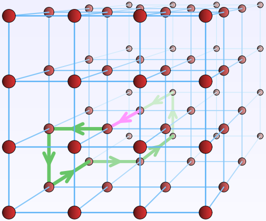


Lattice QCD : dynamical fermions

- In order to have fermion loops, we must keep the Dirac determinant :

$$\mathrm{Det} \sim \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left[\left(g \not{A} \frac{1}{\not{\partial} + m}\right)^n\right]$$

$n = 1$:



Lattice QCD : dynamical fermions

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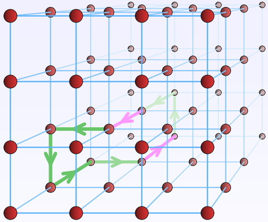
$$\mathrm{Det} \sim \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left[\left(g \not{A} \frac{1}{\not{\partial} + m}\right)^{n}\right]$$

$n = 2 :$



Lattice QCD : dynamical fermions

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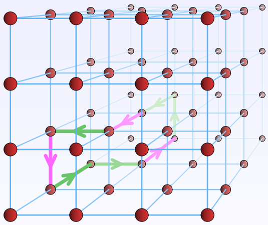
$$\mathrm{Det} \sim \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left[\left(g \not{A} \frac{1}{\not{\partial} + m}\right)^{n}\right]$$

$n = 3$:

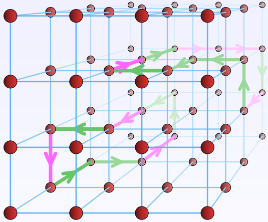


Lattice QCD : dynamical fermions

- In order to have fermion loops, we must keep the Dirac determinant :

$$\mathrm{Det} \sim \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left[\left(g \not{A} \frac{1}{\not{\partial} + m}\right)^{n}\right]$$

Exp : multiple loops



Lattice QCD : dynamical fermions

- As long as there is no baryon chemical potential, the determinant is positive $\Longrightarrow$ include it in the Monte-Carlo sampling

- Complications with fermions :
  - Unphysical degrees of freedom and/or breaking of chiral symmetry
  - When computing correlators between fermionic currents, one needs to invert $\not{D} + m$. Very expensive for light quarks
  - $\mathrm{Det}\,(\not{D} + m)$ not positive definite if $\mu_B > 0$. Monte-Carlo sampling practically impossible

What can be calculated on the lattice?

- Straightforward :
  - Equation of state
  - Quark condensate, Susceptibilities
  - Hadronic masses (but not the widths)
  - Mellin moments of parton distributions
  - Strong coupling constant
- Much more difficult (some dents can be made by brute force) :
  - Transport coefficients
  - Dilepton rates
  - Thermodynamics at non-zero chemical potential
- Impossible (would require a major theoretical breakthrough) :
  - Cross-sections
  - Real-time dynamics of a system

Lattice QCD : calibration

- Lattice calculations give results in units of the lattice spacing a, for the given bare coupling g_0 introduced in the action
- To obtain physical results, one must first compute a known quantity (e.g. some meson mass). This calibrates the lattice spacing :

$$a = \text{some value in fm/c (that depends on } g_0\text{)}$$

Example : computation of a mass

- Find a operator $\mathcal{O}$ that has the right quantum numbers. Ex. : $\mathcal{O} \equiv \overline{\psi}\gamma_5\psi$ for pseudoscalar particles
- Compute the correlator $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle$
- This correlator behaves like :

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \sum_s Z_s \, e^{-iM_s\tau}$$

  s = any state that overlaps with the operator $\mathcal{O}$, with mass M_S

- By fitting the time dependence, one gets the mass of the lightest particle with a given set of quantum numbers
  Note : the precise form of the operator $\mathcal{O}$ does not matter, as long as it overlaps with the particle under consideration
- Caveats :
  - does not give the higher lying states
  - does not give the width

Outline

1. QCD, Deconfinement, Heavy Ion Collisions
2. QCD at finite T, Medium effects, Lattice QCD
3. **Out-of-equilibrium systems, Thermalization**

Out-of-Equilibrium Systems

The naive approach... does not work

- In the Schwinger-Keldysh formalism in momentum space, the propagators contain explicitly the particle distribution
- Naive idea : replace the Bose-Einstein or Fermi-Dirac distributions by non-equilibrium distributions, and use this altered formalism to compute properties of a non-equilibrium quark-gluon plasma

$$
\begin{aligned}
G^0_{++}(p) &= \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi f_p \delta(p^2 - m^2) \\
G^0_{--}(p) &= \frac{-i}{p^2 - m^2 - i\epsilon} + 2\pi f_p \delta(p^2 - m^2) \\
G^0_{+-}(p) &= 2\pi(\theta(-p^0) + f_p)\delta(p^2 - m^2) \\
G^0_{-+}(p) &= 2\pi(\theta(+p^0) + f_p)\delta(p^2 - m^2)
\end{aligned}
$$

Pathologies - Exercise

- The propagators of the Schwinger-Keldysh formalism in momentum space are linear combinations of the distributions

$$\mathrm{P}\,\frac{1}{p^2-m^2}\quad,\qquad \delta(p^2-m^2)$$

- Show that the square of these distributions is ill-defined
- However, some bilinear combinations are well defined :

$$2\left[\mathrm{P}\,\frac{1}{x}\right]\delta(x) = -\frac{d}{dx}\delta(x)$$
$$\pi^2\delta^2(x) - \left[\mathrm{P}\,\frac{1}{x}\right]^2 = \frac{d}{dx}\left[\mathrm{P}\,\frac{1}{x}\right]$$

- For consistency, all the ill-defined products of distributions should cancel when calculating graphs in the Schwinger-Keldysh formalism

Pathologies

- Example : insertion of a self-energy. Consider :

$$= \sum_{\epsilon,\epsilon'=\pm} G^0_{+\epsilon}(p)\,\Sigma_{\epsilon\epsilon'}(p)\,G^0_{\epsilon'+}(p)$$

- This expression contains $\delta^2(p^2-m^2)$ terms (that cannot be combined with others to make finite objects) whose sum is proportional to (for $p^0>0$)

$$2f_{\boldsymbol{p}}(1+f_{\boldsymbol{p}})\Big[\Sigma_{++}+\Sigma_{--}\Big]+(1+2f_{\boldsymbol{p}})\Big[(1+f_{\boldsymbol{p}})\Sigma_{+-}+f_{\boldsymbol{p}}\Sigma_{-+}\Big]$$

- Using the first relation among the $\Sigma_{\epsilon\epsilon'}$'s (which is always true), this coefficient becomes

$$(1+f_{\boldsymbol{p}})\Sigma_{+-}-f_{\boldsymbol{p}}\Sigma_{-+}$$

▷ This is zero only if the KMS identity holds, i.e. if the system is in equilibrium!

Pathologies

- One can learn a bit more by (formally) resumming the self energy on the propagator. Define :

$$\mathbb{G}^0 \equiv \begin{pmatrix} G^0_{++} & G^0_{+-} \\ G^0_{-+} & G^0_{--} \end{pmatrix} \quad , \quad \mathbb{D} \equiv \begin{pmatrix} G^0_F & 0 \\ 0 & G^{0*}_F \end{pmatrix} \quad , \quad \mathbb{S} \equiv \begin{pmatrix} \Sigma_{++} & \Sigma_{+-} \\ \Sigma_{-+} & \Sigma_{--} \end{pmatrix}$$

- **Exercise.** Prove that :

$$\mathbb{G} \equiv \sum_{n=0}^{\infty} \Big[\mathbb{G}^0(-i\mathbb{S})\Big]^n \mathbb{G}^0 = \mathbb{U} \begin{pmatrix} G_F & G_F \widetilde{\Sigma} G^*_F \\ 0 & G^*_F \end{pmatrix} \mathbb{U}$$

with $$G_F(p) \equiv \frac{i}{p^2 - m^2 - \Sigma_F + i\epsilon}$$

and $$\begin{cases} \Sigma_F \equiv \Sigma_{++} + \Sigma_{+-} \\ \widetilde{\Sigma} \equiv \dfrac{1}{1+f_p}\Big[(1+f_p)\Sigma_{+-} - f_p \Sigma_{-+}\Big] \end{cases}$$

- G_F and G^*_F have mirror poles with respect to the real energy axis
  - ▷ pinch singularities if $\mathrm{Im}\,\Sigma_F = 0$

Interpretation

- Compare the bare and resummed propagators :

$$\mathbb{G}^0 = \begin{pmatrix} G_F^0 & \theta(-p^0)(G_F^0 + G_F^{0*}) \\ \theta(+p^0)(G_F^0 + G_F^{0*}) & G_F^{0*} \end{pmatrix} + (G_F^0 + G_F^{0*}) f_p \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbb{G} = \begin{pmatrix} G_F & \theta(-p^0)(G_F + G_F^{*}) \\ \theta(+p^0)(G_F + G_F^{*}) & G_F^{*} \end{pmatrix} + (G_F + G_F^{*}) f_p \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$+ \Big[(1 + f_p)\Sigma_{+-} - f_p \Sigma_{-+}\Big] G_F G_F^* \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- The pinch term gives an equal contribution to the four components of the propagator matrix, exactly like the distribution f_p $\triangleright$ this suggests that this term can be absorbed in a redefinition of f_p

Interpretation

- Strictly speaking, the Schwinger-Keldysh formalism with $t_i = -\infty$ makes sense only in equilibrium

- In fact, the pinch singularities tell us that we are trying to do something a bit stupid :

  We are trying to calculate a certain process taking place at a time x^0 in an out of equilibrium medium, in terms of the particle distribution $f_{\mathbf{p}}$ at the time t_i. This is in principle feasible, but extremely unnatural

  The pinch singularities suggest that it would be much simpler to compute this process in terms of the particle distribution at the time x^0 instead

- By working in coordinate space, we will see that the self-energy resummation amounts - in a certain approximation - to let $f_{\mathbf{p}}$ have a time dependence governed by a Boltzmann equation

Boltzmann Equation

Dyson-Schwinger equations

- In coordinate space, the resummation of the self-energy can be done via the Dyson-Schwinger equations :

$$G(x,y) = G^0(x,y) + \int_{\mathcal{C}} d^4u d^4v \; G^0(x,u)\Big(-i\Sigma(u,v)\Big)G(v,y)$$

$$G(x,y) = G^0(x,y) + \int_{\mathcal{C}} d^4u d^4v \; G(x,u)\Big(-i\Sigma(u,v)\Big)G^0(v,y)$$

- Apply $\square_x + m^2$ to the first equation :

$$(\square_x + m^2)G(x,y) = -i\delta_c(x-y) - \int_{\mathcal{C}} d^4v \; \Sigma(x,v) \; G(v,y)$$

- Similarly,

$$(\square_y + m^2)G(x,y) = -i\delta_c(x-y) - \int_{\mathcal{C}} d^4v \; G(x,v) \; \Sigma(v,y)$$

Wigner transform

- Out of equilibrium, 2-point functions depend separately on their two arguments (in equilibrium they depend only on the difference $x - y$)

- It is useful to perform a Fourier transform with respect to the difference $s \equiv x - y$. Wigner transform of $F(x, y)$:

$$F(X, p) \equiv \int d^4 s \; e^{ip\cdot s} \; F(X + \frac{s}{2}, X - \frac{s}{2})$$

- Derivatives with respect to x and y can be written in terms of derivatives with respect to X and s :

$$\partial_x = \frac{1}{2}\partial_X + \partial_s \quad , \quad \partial_y = \frac{1}{2}\partial_X - \partial_s$$

$$\square_x = \frac{1}{4}\square_X + \partial_X \cdot \partial_s + \square_s \quad , \quad \square_y = \frac{1}{4}\square_X - \partial_X \cdot \partial_s + \square_s$$

Wigner transform - Exercise

- Wigner transform of a convolution. Consider :

$$H(x,y) \equiv \int d^4z\, F(x,z)\, G(z,y)$$

- Prove that :

$$H(X,p) = F(X,p)\; e^{\frac{i}{2}\left[\overleftarrow{\partial}_X \overrightarrow{\partial}_p - \overrightarrow{\partial}_X \overleftarrow{\partial}_p\right]}\; G(X,p)$$

- By expanding the exponential, one gets the gradient expansion of the Wigner transform of the convolution product

Gradient expansion

- The derivatives with respect to X ($\partial_x, \square_x$) characterize the space and time scales over which the particle distribution changes significantly
- We assume that these scales are much larger than the De Broglie wavelength of the particles, i.e. that $\partial_x \ll p, \square_x \ll p^2$
- Note : typically, ∂_x is at most of the order of the inverse transport mean free path, i.e. $g^4 T$
- As we shall see, the relevant self-energy in transport phenomena is of order $g^4 T^2$, while the typical particle momentum is of order T

  $\triangleright$ it is sufficient to expand the convolution product in the r.h.s. to zeroth order in gradients

Gradient expansion

- By taking the difference of the Dyson-Schwinger equations w.r.t. x and y, and by breaking it down into its $\pm$ components, one finds

$$-2ip\cdot\partial_X\left(G_{+-}(X,p)-G_{-+}(X,p)\right)=0$$
$$-2ip\cdot\partial_X\left(G_{+-}(X,p)+G_{-+}(X,p)\right)=2\Big[G_{-+}\Sigma_{+-}-G_{+-}\Sigma_{-+}\Big]$$

- Quasi-particle ansatz : by analogy with the free theory, one assumes that (for $p^0>0$)

$$G_{-+}(X,p)=(1+f(X,p))\rho(X,p)$$
$$G_{+-}(X,p)=f(X,p)\rho(X,p)$$

where $\rho(X,p)\equiv G_{-+}(X,p)-G_{+-}(X,p)$

- This assumption is valid when the quasi-particles are long-lived. This requires a small coupling and a moderate density

Boltzmann equation

Boltzmann equation :

$$\left[\partial_t + \vec{v}_{\mathbf{p}} \cdot \vec{\nabla}_{\vec{x}}\right] f(X,p) = \frac{i}{2E_{\mathbf{p}}}\left[(1+f(X,p))\Sigma_{+-} - f(X,p)\Sigma_{-+}\right]$$

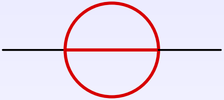
where $\vec{v}_{\mathbf{p}} \equiv \vec{\mathbf{p}}/E_{\mathbf{p}}$

- In the r.h.s (collision term), we see the same combination as in the KMS condition $\triangleright$ it is zero in equilibrium
- The collision term is a (spatially local) functional of the particle distribution $f(X,p)$ $\triangleright$ the Boltzmann equation is an approximation of the Dyson-Schwinger equations in which the degrees of freedom are on-shell particles
- The combination $\partial_t + \vec{v}_{\mathbf{p}} \cdot \vec{\nabla}_{\vec{x}}$ is the transport derivative

  It is zero on any function whose t and $\vec{x}$ dependence arise only in the combination $\vec{x} - \vec{v}_{\mathbf{p}}\, t$

- Consider a scalar theory with a $\lambda\phi^4$ interaction

- Show that the first non-zero contribution to the collision term arises at 2-loops, in the diagram



- Calculate the corresponding collision term, and show that it is given by

$$\frac{\lambda^2}{4E_p}\int\frac{d^3\vec{\mathbf{p}}_1}{(2\pi)^3 2E_1}\int\frac{d^3\vec{\mathbf{p}}_2}{(2\pi)^3 2E_2}\int\frac{d^3\vec{\mathbf{p}}_3}{(2\pi)^3 2E_3}(2\pi)^4\delta(p-p_1-p_2-p_3)$$
$$\times\Big[f(p_1)f(p_2)(1+f(p_3))(1+f(p))-f(p_3)f(p)(1+f(p_1))(1+f(p_2))\Big]$$

(General structure : Gain term – Loss term)

Boltzmann-Vlasov equation

- Our derivation must be slightly modified when the self-energy $\Sigma(u,v)$ contains a local part :

$$\Sigma(u,v) = \Phi(u)\delta_c(u-v) + \Pi(u,v)$$

- In the derivation of the Boltzmann equation, one needs the Wigner transform of

$$\Phi(y)G(x,y) - \Phi(x)G(x,y)$$

Exercise : show that to lowest order in the gradient expansion, this Wigner transform is

$$i\partial_x\Phi(X)\cdot\partial_p G(X,p)$$

- The modified Boltzmann equation reads :

$$\left[\partial_t + \vec{v}_{\mathbf{p}}\cdot\vec{\nabla}_{\vec{x}}\right] f + \underline{\frac{1}{2E_{\mathbf{p}}}\partial_x\Phi\cdot\partial_p f} = \frac{i}{2E_{\mathbf{p}}}\left[(1+f)\Sigma_{+-} - f\Sigma_{-+}\right]$$

The trouble with large occupation numbers

- Immediately after the collision of two heavy nuclei, the gluon occupation number $f(\mathbf{p})$ is large, of order $1/g^2$
- If $f(\mathbf{p}) \sim g^{-2}$, several issues arise :
  - It is not possible to truncate the collision term
  - Quasi-particles do not exist (their decay width is comparable to their mass)
  - Gauge fields are large ($\sim g^{-1}$)
- In principle, lattice QCD has no problem with large fields. BUT : it is an Euclidean method, not suited for following the real-time dynamics of an out-of-equilibrium system

Classical Statistical Field Theory

Analogous approximation in Quantum Mechanics

- Consider the von Neumann equation for the density operator :

$$\frac{\partial\widehat{\rho}_\tau}{\partial\tau} = i\hbar\,\big[\widehat{H},\widehat{\rho}_\tau\big] \qquad (**)$$

- Introduce the Wigner transforms :

$$\begin{aligned} W_\tau(x,\mathbf{p}) &\equiv \int ds\; e^{i\mathbf{p}\cdot\mathbf{s}}\,\big\langle x+\frac{\mathbf{s}}{2}\big|\widehat{\rho}_\tau\big|x-\frac{\mathbf{s}}{2}\big\rangle \\ \mathcal{H}(x,\mathbf{p}) &\equiv \int ds\; e^{i\mathbf{p}\cdot\mathbf{s}}\,\big\langle x+\frac{\mathbf{s}}{2}\big|\widehat{H}\big|x-\frac{\mathbf{s}}{2}\big\rangle \quad \text{(classical Hamiltonian)} \end{aligned}$$

- Then, (**) is equivalent to

$$\begin{aligned} \frac{\partial W_\tau}{\partial\tau} &= \mathcal{H}(x,\mathbf{p})\,\frac{2}{i\hbar}\,\sin\left(\frac{i\hbar}{2}\big(\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_x-\overleftarrow{\partial}_x\overrightarrow{\partial}_{\mathbf{p}}\big)\right)\,W_\tau(x,\mathbf{p}) \\ &= \underbrace{\big\{\mathcal{H},W_\tau\big\}}_{\text{Poisson bracket}} + \mathcal{O}(\hbar^2) \end{aligned}$$

Analogous approximation in Quantum Mechanics

- Approximating the full right hand side by the Poisson bracket is the same as solving classical equations of motion instead of the full quantum evolution. This leads to an $\mathcal{O}(\hbar^2)$ error

- There is also an $\hbar$ dependence coming from the initial state. In Quantum Mechanics, the uncertainty principle states that $\Delta\mathbf{x}\cdot\Delta\mathbf{p}\geq\hbar$. This implies that the initial Wigner distribution $W_{\tau=0}(\mathbf{x},\mathbf{p})$ cannot be localized at a single point in phase-space – it must have a width of extension $\hbar$ (at least)

- All the $\mathcal{O}(\hbar)$ effects can be accounted for by a Gaussian initial distribution $W_{\tau=0}(\mathbf{x},\mathbf{p})$

- The initial Gaussian distribution can be sampled by a Monte-Carlo. For each initial $(\vec{\mathbf{x}},\vec{\mathbf{p}})$, solve the classical equation of motion up to the time of interest

Classical Hamiltonian lattice QCD

- Choose a variable that you call “time”
  ($\tau = \sqrt{t^2 - z^2}$ in a high energy collision)
- Conjuguate momenta : $E \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\tau A)}$. Hamiltonian : $\mathcal{H} = EA - \mathcal{L}$
- Classical equations of motion :

$$\partial_\tau A = \frac{\partial \mathcal{H}}{\partial E} \quad , \quad \partial_\tau E = -\frac{\partial \mathcal{H}}{\partial A}$$

- Lattice setup :
  - Discretize space on a 3-dim cubic lattice. Keep time continuous
  - The gauge potential A^i are described as link variables living on the edges of the lattice. The A^τ component lives on the nodes (but in practice, one ignores it altogether by choosing the $A^\tau = 0$ gauge)
  - The electrical fields E^i live on the nodes of the lattice

Classical Hamiltonian lattice QCD

- Hamiltonian in $A^\tau = 0$ gauge :

$$\mathcal{H} = \sum_{\vec{x};i} \frac{E^i(x)E^i(x)}{2} - \frac{6}{g_0^2} \sum_{\vec{x};ij} 1 - \frac{1}{3}\mathrm{Re}\,\mathrm{Tr}\,\underbrace{\left(U_i(x)U_j(x+\hat{\imath})U_i^\dagger(x+\hat{\jmath})U_j^\dagger(x)\right)}_{\text{plaquette at the point } \vec{x} \text{ in the } ij \text{ plane}}$$

- Properties :
  - Invariant under the residual gauge transformations that preserves $A^\tau = 0$ (i.e. time independent gauge transformations)
  - Hamilton equations $\Leftrightarrow$ lattice classical Yang-Mills equations

- The Hamilton equations on the lattice form a (large) set of ordinary differential equations, that can e.g. be solved with the leapfrog algorithm

Initial condition

- The typical application of this method is the Color Glass Condensate description of heavy ion collisions :

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{\text{strong color currents}} A_\mu$$

- One would like to compute expectation values such as

$$\big\langle \mathcal{O}_{\tau,\vec{x}} \big\rangle \equiv \big\langle 0_{\rm in} \big| \mathcal{O}[A(\tau,\vec{x}), E(\tau,\vec{x})] \big| 0_{\rm in} \big\rangle$$

Leading Order in g^2 :

$$\big\langle \mathcal{O}_{\tau,\vec{x}} \big\rangle = \mathcal{O}[\mathcal{A}_{\rm cl}(\tau,\vec{x}), \mathcal{E}_{\rm cl}(\tau,\vec{x})]$$

where $\mathcal{A}_{\rm cl}, \mathcal{E}_{\rm cl}$ are the solutions of the classical Yang-Mills equations such that

$$\lim_{\tau\to-\infty} \mathcal{A}_{\rm cl}, \mathcal{E}_{\rm cl} = 0$$

Fluctuating initial conditions

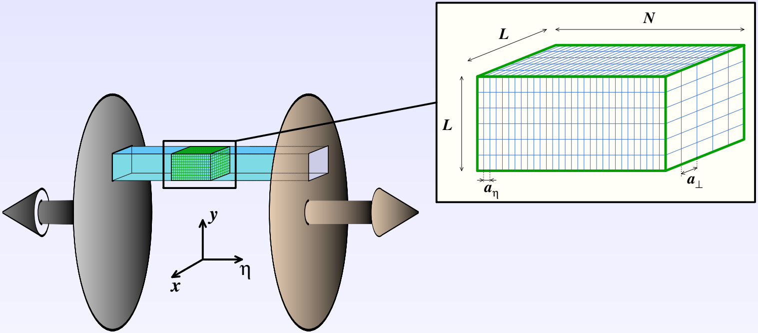
- By having a distribution of initial conditions instead of a single one, one can go beyond leading order
- If one averages over Gaussian fluctuations of the initial condition, there is a (unique) choice of the fluctuations for which :
  - One gets also the correct Next-to-Leading Order result
  - One resums an infinite class of loop corrections
- To obtain the variance of this Gaussian distribution, one must perform a 1-loop calculation

Application to heavy-ion collisions [FG, T. Epelbaum (2013)]

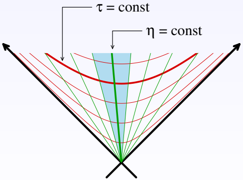
Main steps

1. Solve (analytically) the equation of motion of small perturbations on top of the classical background, in order to obtain the correct spectrum of Gaussian fluctuations
2. Solve numerically the Yang-Mills equations on a lattice in 3+1 dimensions
3. Do a Monte-Carlo sampling of the fluctuating initial conditions

Discretization of the expanding volume



- Comoving coordinates : $\tau, \eta, \boldsymbol{x}_\perp$
- Only a small volume is simulated
  + periodic boundary conditions
- $L^2 \times N$ lattice with $L \sim 64-100$, $N \sim 128-200$

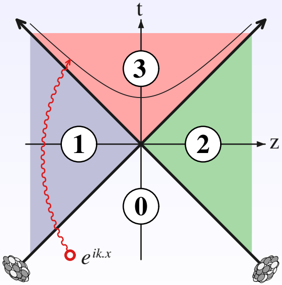


Gaussian spectrum of fluctuations

Expression of the variance (from 1-loop considerations)

$$\Gamma_2(u,v) = \int_{\text{modes } k} a_k(u)a_k^*(v)$$

$$\mathcal{D}_\mu\mathcal{D}^\mu a_k^\nu - \mathcal{D}_\mu\mathcal{D}^\nu a_k^\mu + ig\,\mathcal{F}_\mu{}^\nu a_k^\mu = 0 \quad , \quad \lim_{x^0\to-\infty} a_k(x) \sim e^{ik\cdot x}$$



0. $\mathcal{A}^\mu = 0$, trivial

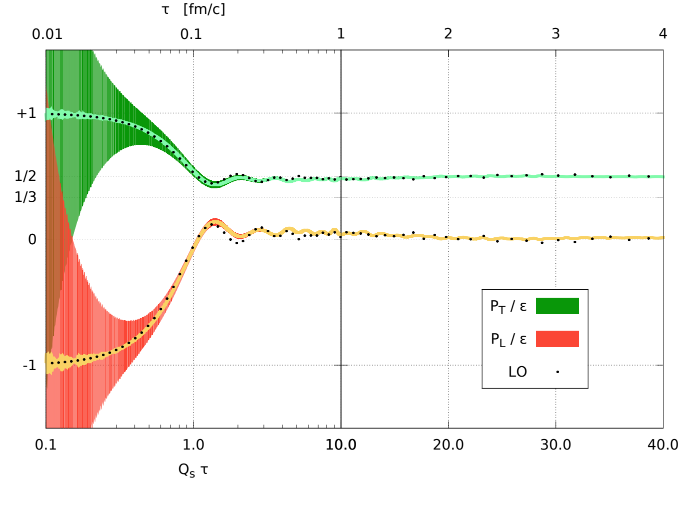
1,2. $\mathcal{A}^\mu =$ pure gauge, analytical solution

3. $\mathcal{A}^\mu$ non-perturbative, lowest order expansion in $Q_s\tau$

- We need the fluctuations in Fock-Schwinger gauge $x^+a^- + x^-a^+ = 0$
- Beware of the light-cone crossings, since $\mathcal{F}^{\mu\nu} = \infty$ there

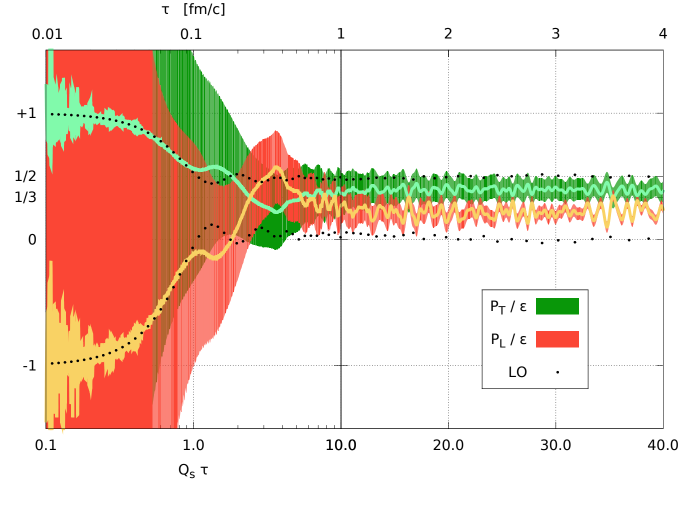
Isotropization

$g = 0.1$



Isotropization

$g = 0.5$



Thank You!