# Towards small $x$

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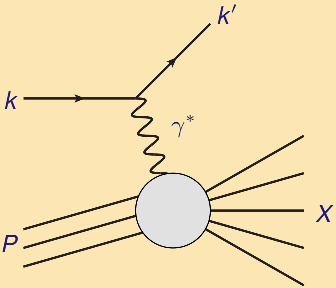
Hard Probes summer school, Cape Town, Nov 2013

# 1 DIS in the dipole picture

# DIS kinematics, high energy=small $x$




$$
\begin{aligned}
s &= (k+P)^2 \\
q &= k - k' \quad q^2 \equiv -Q^2 \\
W^2 &= (P+q)^2 \\
\nu &= P \cdot q / m_N \\
x &= \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2\nu m_N} = \frac{Q^2}{W^2 + Q^2 - m_N^2} \\
y &= \frac{2P\cdot q}{2P\cdot k} = \frac{W^2 + Q^2 - m_N^2}{s m_N^2}
\end{aligned}
$$

**High energy limit is $x \longrightarrow 0$**

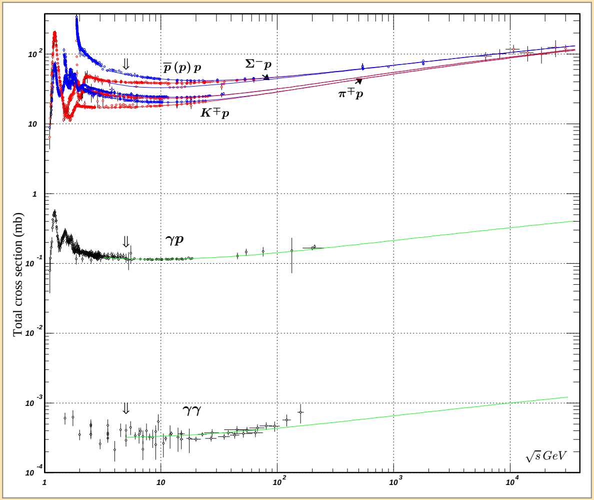
- This is when $W^2 \to \infty$ ; $\nu \to \infty$ ;
  i.e. the virtual photon-target c.m.s. energy is high.
- Now $Q^2$ is "fixed".
  In DGLAP the limit is $x$ fixed, $Q^2$ large (large transverse momentum)

I want to convince you that the $\gamma^*$ is the theorist's favorite hadron!

# Cross sections vs. energy

$\gamma$ scattering behaves just like $p$ scattering — apart from extra $\frac{1}{137}$

The same should be true for $\gamma^*$



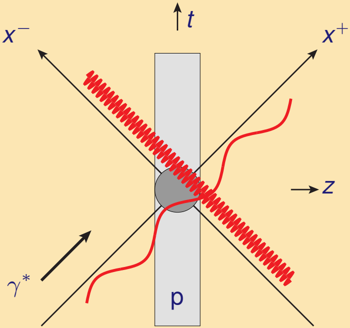
# Kinematical variables in TRF

Light cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}}(t \pm z)$ (Note boldface **x** is 2d transverse)

$$
\begin{aligned}
P^{\mu} &= (\overset{0}{m}, \overset{\perp}{\mathbf{0}}, \overset{z}{0}) \implies (\overset{+}{m/\sqrt{2}}, \overset{-}{m/\sqrt{2}}, \overset{\perp}{\mathbf{0}}) \\
q^{\mu} &= (\overset{0}{\nu}, \overset{\perp}{\mathbf{0}}, \overset{z}{\sqrt{\nu^2 + Q^2}}) \implies (\overset{+}{q^+}, \overset{-}{-Q^2/(2q^+)}, \overset{\perp}{\mathbf{0}})
\end{aligned}
$$

High energy: $q^+ \approx \sqrt{2}\nu$ **big**

Look at $\gamma^*$ wavefunction $e^{-i(q^+x^- + q^-x^+)}$

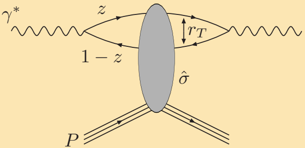



- Very accurate resolution in $x^-$
- No resolution in $x^+$
  Scattering instantaneous in $x^+$ compared to natural timescale of $\gamma^*$

In particular $\gamma^*$ cannot change into a hadronic final state **inside** proton; it has to fluctuate into hadrons before.

# DIS in dipole picture

Simplest hadronic state in the interacting $\gamma^*$ state: quark-antiquark dipole.



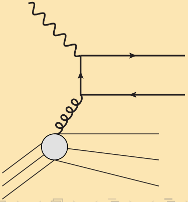

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}\,\mathrm{d}z \left|\psi^{\gamma^* \to q\bar{q}}(\mathbf{r}, z)_{T,L}\right|^2 2\mathrm{Im}\mathcal{A}$$

**High energy:** we assume (lifetime/timescale) factorization between

- $\left|\psi^{\gamma^* \to q\bar{q}}(\mathbf{r}, z)_{T,L}\right|^2$: probability for photon to fluctuate into $\bar{q}q$
- $2\mathrm{Im}\mathcal{A}$ imaginary part of the forward elastic scattering amplitude, i.e. the total cross section; optical theorem

Same process in the IMF would look like this

- Formally higher order (NLO DIS)
- Dominates at small $x$ because $xg(x, Q^2)$ is large
- Does not describe valence quarks



## Virtual photon wavefunction $\psi^{\gamma^* \to q\bar{q}}$

The concept makes sense in the framework of
**L**ight **C**one **P**erturbation **T**heory: (No time to go very far here)

### Outline of LCPT calculation

- Idea: know free particle Fock states: $|\gamma^*\rangle_0, \quad |q\bar{q}\rangle_0, \quad |q\bar{q}g\rangle_0$ etc.
- **Interacting** states are superpositions of these:

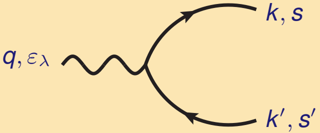
$$|\gamma^*\rangle = (1 + \ldots)|\gamma^*\rangle_0 + \psi^{\gamma^* \to q\bar{q}} \otimes |q\bar{q}\rangle_0 + \psi^{\gamma^* \to q\bar{q}g} \otimes |q\bar{q}g\rangle_0 + \ldots$$

- QM perturbation theory: ground state $|0\rangle$ wavefunction correction is

$$\sum_n \frac{\langle n| \hat{V} |0\rangle}{E_n - E_0} |n\rangle$$

  - Here $1/\Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to $n$
- In LCPT, "energy" is $k^-$
- Matrix elements $\langle n| \hat{V} |0\rangle$ are vertices in Feynman rules

# Calculating $\psi^{\gamma^* \to q\bar{q}}$




Need two things to calculate $\psi^{\gamma^* \to q\bar{q}}$

- Matrix element

$$\sim e\bar{u}_s(k)\not{\varepsilon}_\lambda v_{s'}(k') \quad ; \quad s, s' = \pm\frac{1}{2}; \quad \lambda = 0 = L, \quad \lambda = \pm 1 = T$$

- Energy denominator $(q^- - k^- - k'^-)^{-1}$

$$= -\left(\frac{Q^2}{2q^+} + \frac{\mathbf{k}^2 + m^2}{2zq^+} + \frac{\mathbf{k}^2 + m^2}{2(1-z)q^+}\right) = \frac{-2q^+ z(1-z)}{\underbrace{Q^2 z(1-z) + m^2}_{\equiv \varepsilon^2} + \mathbf{k}^2}$$

Fourier-transform $\mathbf{k} \to \mathbf{r}$, sum over spins; result is

$$\left|\psi_T^{\gamma^* \to q\bar{q}}\right|^2 = \frac{\alpha_{\text{e.m.}}}{2\pi^2} N_c e_f \left(\left[z^2 + (1-z)^2\right] K_1^2(\varepsilon r) + m_f^2 K_0^2(\varepsilon r)\right)$$

$$\left|\psi_L^{\gamma^* \to q\bar{q}}\right|^2 = \frac{\alpha_{\text{e.m.}}}{2\pi^2} N_c e_f 4 Q^2 z^2 (1-z)^2 K_0^2(\varepsilon r)$$

# DIS dipole frame: summary

- Picture DIS as $\gamma^*$ scattering on target
- At high energy (in TRF) $\gamma^*$ fluctuates into $q\bar{q}$

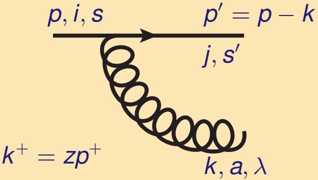
$$\sigma^{\gamma^* p}_{T,L} = \int \mathrm{d}^2\mathbf{r}\,\mathrm{d}z \left|\psi^{\gamma^*\to q\bar{q}}(r,z)_{T,L}\right|^2 2\mathrm{Im}\mathcal{A}$$

$$\left|\psi^{\gamma^*\to q\bar{q}}(r,z)_{T,L}\right|^2 \sim \exp\left\{\sqrt{z(1-z)}Qr\right\}$$

- Typical dipole size: $r \sim 1/Q$
- Used optical theorem: $2\mathrm{Im}\mathcal{A}$ is total cross section
  - can also take $|\mathcal{A}|^2$ : elastic scattering (diffractive DIS)
- We are assuming that fixed-size dipoles are the basis that diagonalizes the imaginary part of the $T$-matrix
  - This makes sense in an eikonal approximation for the scattering
  - In general: high energy/eikonal approximation: particles fly through target at fixed $\mathbf{x}$; does not imply zero momentum transfer!

# 2 Balitsky-Kovchegov equation

# What happens if one radiates a gluon?




Light cone wavefunction

$$\psi^{q\to qg}(z,\mathbf{k}) = \frac{\sqrt{p^+}}{p^- - \frac{\mathbf{k}^t}{2k^+} - \frac{\mathbf{p}'^2}{2p'^+)}} \times \frac{\bar{u}_{s'}(p')}{\sqrt{(2\pi)^3 2p'^+}} \frac{t^a_{ij} g \not{\varepsilon}(k)}{\sqrt{(2\pi)^3 2k^+}} \frac{u_s(p)}{\sqrt{(2\pi)^3 2p^+}}$$

Matrix elements from Pauli `hep-ph/0103106`

This is simple in the **soft** limit $z \to 0$:

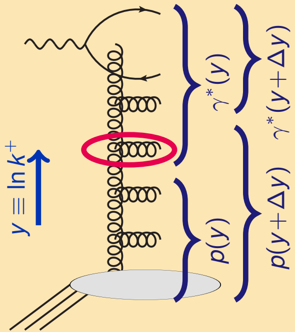
$$\psi^{q\to qg}(z,\mathbf{k}) = -\frac{g}{2\pi^{3/2}} t^a_{ij} \frac{1}{\sqrt{z}} \frac{\varepsilon\cdot\mathbf{k}}{\mathbf{k}^2}\delta_{s,s'} \qquad |\psi|^2 \sim \frac{\mathrm{d}P}{\mathrm{d}z\,\mathrm{d}^2\mathbf{k}} \sim \frac{1}{z}\frac{1}{\mathbf{k}^2} \qquad \left(\sum_{\lambda=\pm1} \varepsilon_i \varepsilon_j^* = g_{ij}\right)$$

Typical gauge theory logarithmic divergences in emission probability:

*soft* $\frac{\mathrm{d}z}{z}$

*collinear* $\frac{\mathrm{d}^2\mathbf{k}}{\mathbf{k}^2}$

# Soft gluons and large logs, idea of RGE




- Emitted gluons have $z$ between 1 and $x$: each gluon contributes $\sim \alpha_s \ln 1/x$
- For $x$ small $\alpha_s \ln 1/x \sim 1 \implies$ all $n$ gluon emissions contribute same $\implies$ resum
- Cone by Renormalization Group Equation

**Is the gluon at $y$ a part of $\gamma^*$ or of $p$?**

You have to decide!
Physical cross section is the same.

$$\sigma^{\gamma^* p} = \overbrace{\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_y \otimes 2\mathrm{Im}\mathcal{A}_y^{q\bar{q}p} + \left|\psi^{\gamma^*\to q\bar{q}g}\right|^2_y \otimes 2\mathrm{Im}\mathcal{A}_y^{q\bar{q}gp} + \dots}^{\text{gluons up to } y \text{ are part of proton}}$$

$$= \underbrace{\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_{y+\Delta y} \otimes 2\mathrm{Im}\mathcal{A}_{y+\Delta y}^{q\bar{q}p} + \left|\psi^{\gamma^*\to q\bar{q}g}\right|^2_{y+\Delta y} \otimes 2\mathrm{Im}\mathcal{A}_{y+\Delta y}^{q\bar{q}gp} + \dots}_{\text{gluons up to } y+\Delta y \text{ are part of proton}}$$

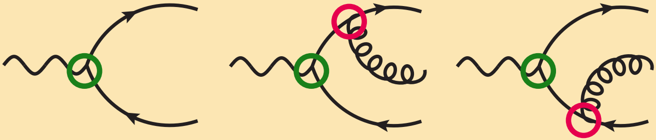
Can calculate $\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_y$'s $\implies$ get differential equation for unknown $\mathcal{A}$

## Gluon emission from coordinate space dipole

Let's put this idea into practice. We will

- Calculate $\psi^{\gamma^* \to q\bar{q}g}(z)$
- Take soft gluon limit $z \to 0$
- Reabsorb the gluon to become a part of the target
- Get evolution equation for $q\bar{q}$ cross section

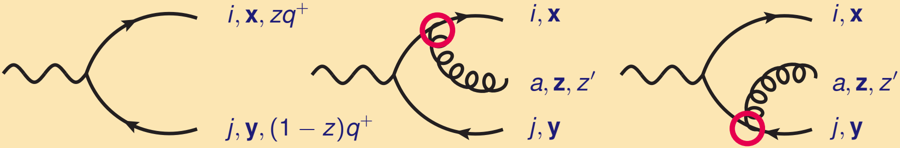
We need:



We can do this with $\psi^{\gamma^* \to q\bar{q}}$ we already know and and coordinate space

$$\psi^{q \to qg}(z, \mathbf{r}) = \int \frac{d^2\mathbf{k}}{\sqrt{(2\pi)^3}} e^{i\mathbf{k}\cdot\mathbf{r}} \psi^{q\to qg}(z, \mathbf{k}) = -i \frac{g}{2\pi^{3/2}} t^a_{ij} \frac{1}{\sqrt{z}} \frac{\varepsilon \cdot \mathbf{r}}{\mathbf{r}^2} \delta_{s,s'}$$

# Gluon emission from coordinate space dipole




$$\mathbf{r} = \mathbf{x} - \mathbf{y} \quad \mathbf{r}' = \mathbf{x} - \mathbf{z} \quad \mathbf{z} - \mathbf{y} = \mathbf{r} - \mathbf{r}' \qquad \psi^{q \to qg}(z')$$

$$|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle + \int_{z,\mathbf{r}} \frac{C(\mathbf{r})}{\sqrt{N_c}} \psi^{\gamma^* \to q\bar{q}}(z, \mathbf{r})\, |q_i(\mathbf{x})\bar{q}_j(\mathbf{y})\rangle$$

$$+ \int_{z,\mathbf{r},z',\mathbf{r}'} \frac{1}{\sqrt{N_c}} \psi^{\gamma^* \to q\bar{q}}(z, \mathbf{r}) \frac{-ig}{2\pi^{3/2}} \frac{t^a_{ij}}{\sqrt{z'}} \left[ \frac{(\mathbf{x}-\mathbf{z})\cdot\varepsilon}{(\mathbf{x}-\mathbf{z})^2} - \frac{(\mathbf{y}-\mathbf{z})\cdot\varepsilon}{(\mathbf{y}-\mathbf{z})^2} \right] |q_i(\mathbf{x})\bar{q}_j(\mathbf{y})g_a(\mathbf{z})\rangle$$

Adjust coefficient of $q\bar{q}$-state to keep wavefunction normalized:

$$|C(\mathbf{r})|^2 = 1 - \frac{g^2}{4\pi^3} \frac{1}{N_c} t^a_{ij} t^a_{ji} \int \frac{\mathrm{d}z'}{z'} \int \mathrm{d}^2\mathbf{r}' \sigma_{\lambda=\pm 1} \left| \frac{(\mathbf{x}-\mathbf{z})\cdot\varepsilon_\lambda}{(\mathbf{x}-\mathbf{z})^2} - \frac{(\mathbf{y}-\mathbf{z})\cdot\varepsilon_\lambda}{(\mathbf{y}-\mathbf{z})^2} \right|^2$$

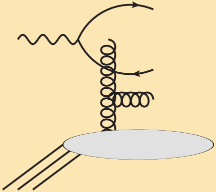
$$= 1 - \frac{\alpha_s}{\pi^2} \frac{N_c^2 - 1}{2N_c} \Delta y \int \mathrm{d}^2\mathbf{r}' \frac{\mathbf{r}^2}{\mathbf{r}'^2(\mathbf{r}-\mathbf{r}')^2}$$

$$\sum_{\lambda=\pm 1} \varepsilon_i^{(\lambda)} \varepsilon_j^{(\lambda)*} = g_{ij}$$

# Crucial step: move the gluon to the target

Scattering amplitude is $\mathrm{Im}\mathcal{A}(\mathbf{r}) = \int \mathrm{d}^2\mathbf{b}\mathcal{N}(\mathbf{b},\mathbf{r})$.
We want equality between scatterng amplitudes with gluon in different place:

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c{}^2-1}{2N_c}\int_y^{y+\Delta y} \mathrm{d}\ln 1/z' \int \mathrm{d}^2\mathbf{r}' \frac{\mathbf{r}^2}{\mathbf{r}'^2(\mathbf{r}-\mathbf{r}')^2}\left[\mathcal{N}_{q\bar{q}g}^{\ln 1/z'} - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}\right]$$



Dipole scattering on new target $\mathcal{N}_{q\bar{q}}^{y+\Delta y}$ is

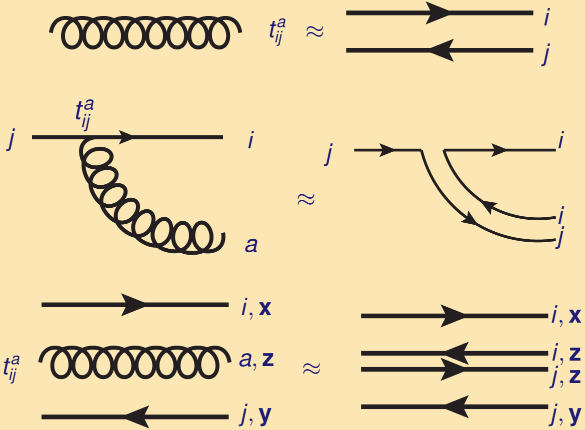
- Dipole scattering off original target $\mathcal{N}_{q\bar{q}}^{y}$
- Dipole emits a gluon into rapidity interval $[y, y+\Delta y]$, which scatters off target
- Normalization of original dipole is corrected (There are now less dipoles in $\gamma^*$)

## Almost there

We are looking for an equation for $\mathcal{N}_{q\bar{q}}$: but enocuntered new quantity $\mathcal{N}_{q\bar{q}g}$, which needs to be related to $\mathcal{N}_{q\bar{q}}$. Will do this in the large $N_c$ approximation

# Gluon at large $N_c$

- At large $N_c$ $\Longrightarrow$ gluon = $q\bar{q}$ pair (not dipole!)
- $N_c^2 - 1$ gluon colors $\approx N_c^2$ quark-antiquark pair colors.
- Had $|q(\mathbf{x})\bar{q}(\mathbf{y})g(\mathbf{z})\rangle$
- Approximate by $|q(\mathbf{x})\bar{q}(\mathbf{z})q(\mathbf{z})\bar{q}(\mathbf{y})\rangle$




Now, instead of $\mathcal{N}_{q\bar{q}g}$, we need $\mathcal{N}_{q\bar{q}q\bar{q}}$;
amplitude for simultaneous scattering of two dipoles.

# Two gluon scattering amplitude

- $\mathcal{N}$ is really **scattering probability**;
- $S = 1 - \mathcal{N}$ is probability **not to scatter**

For two dipoles:

- No scattering: neither dipole scatters
  $\Longrightarrow S_{q\bar{q}q\bar{q}} = S_{q\bar{q}} S_{q\bar{q}}$
- Scattering probability $\mathcal{N}_{q\bar{q}q\bar{q}} = 1 - S_{q\bar{q}q\bar{q}} = 1 - (1 - \mathcal{N}_{q\bar{q}})(1 - \mathcal{N}_{q\bar{q}})$

Thus we end up with the approximation:

$$\mathcal{N}(q(\mathbf{x})\bar{q}(\mathbf{y})g(\mathbf{z})) \approx \mathcal{N}(q(\mathbf{x})\bar{q}(\mathbf{z})) + \mathcal{N}(q(\mathbf{z})\bar{q}(\mathbf{y})) - \mathcal{N}(q(\mathbf{x})\bar{q}(\mathbf{z}))\mathcal{N}(q(\mathbf{z})\bar{q}(\mathbf{y}))$$

and our equation is

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c{}^2 - 1}{2N_c}\int_y^{y+\Delta y} \mathrm{d}\ln 1/z' \int \mathrm{d}^2\mathbf{z}\, \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}$$

$$\times\left[\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x},\mathbf{z}) + \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z},\mathbf{y}) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x},\mathbf{z})\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z},\mathbf{y}) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x},\mathbf{y})\right]$$

Differentially for infinitesimal $\Delta y$, and with large $N_c$

$$\partial_y \mathcal{N}(\mathbf{r}) = \frac{\alpha_s N_c}{2\pi^2}\int \mathrm{d}^2\mathbf{r}' \frac{\mathbf{r}^2}{\mathbf{r}'^2(\mathbf{r}'-\mathbf{r})^2}\left[\mathcal{N}(\mathbf{r}') + \mathcal{N}(\mathbf{r}-\mathbf{r}') - \mathcal{N}(\mathbf{r}')\mathcal{N}(\mathbf{r}-\mathbf{r}') - \mathcal{N}(\mathbf{r})\right]$$

# Summary

## Balitsky-Kovchegov equation ($\sim$1995)

$$\partial_y \mathcal{N}(\mathbf{r}) = \frac{\alpha_s N_c}{2\pi^2} \int d^2\mathbf{r}' \frac{\mathbf{r}^2}{\mathbf{r}'^2(\mathbf{r}'-\mathbf{r})^2} \times \left[\mathcal{N}(\mathbf{r}') + \mathcal{N}(\mathbf{r}-\mathbf{r}') - \mathcal{N}(\mathbf{r}')\mathcal{N}(\mathbf{r}-\mathbf{r}') - \mathcal{N}(\mathbf{r})\right]$$

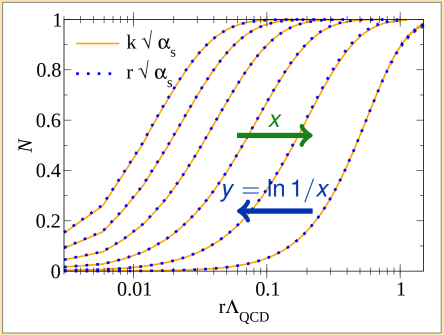
This is the basic tool of modern small-$x$ physics.

- Given initial condition $\mathcal{N}(\mathbf{r})$ at $y = y_0$ the equation predicts the scattering amplitude at larger $y$ = smaller $x$ = higher $\sqrt{s}$.
- Drop nonlinear term: BFKL equation
- Divergences at $\mathbf{r}' \to 0$ and $\mathbf{r}' \to \mathbf{r}$ regulated because $\mathcal{N}(0) = 0$ due to color neutrality.
- Enforces black disk limit (unitraity) $\mathcal{N} < 1$
- For practical work coupling $\alpha_s$ should depend on distance: some combination of $\mathbf{r}, \mathbf{r}', \mathbf{r}-\mathbf{r}'$

# What the solution of BK looks like

The equation can be solved numerically

- Small dipoles $r \lesssim 1/Q_s$ scatter very little
  At $r = 0$ color neutral system, should not scatter by the strong interaction!
- Large dipoles $r \gtrsim 1/Q_s$ scatter with probability almost one, but not more. **Saturation**




(Actually cheating, this plot is a solution of JIMWLK, which generalizes BK)

Remember, for the DIS $F_2, F_L$ convolute this with the (known) $\gamma^*$ wavefunction.

$$\sigma^{\gamma^* p}_{T,L} = \int \mathrm{d}^2\mathbf{b}\, \mathrm{d}^2\mathbf{r}\, \mathrm{d}z \left|\psi^{\gamma^* \to q\bar{q}}(r,z)_{T,L}\right|^2 2\mathcal{N}(\mathbf{r},\mathbf{b},x)$$

Fits HERA data ($x < 0.01$ $Q^2$ moderate) extremely well

($b$-dependence modeled with varying degrees of sophistication)

# 3 Eikonal propagation in target color field

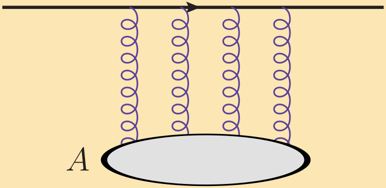
# What is the target made of?

- So far we have not specified anything about the degrees of freedom in the target.
- We will srgue that at high energy the target consists dominantly of gluons
  - We know that at small $x$ the gluon distribution is larger than the quark one.
  - BK equation builds up the target by adding gluons to it.

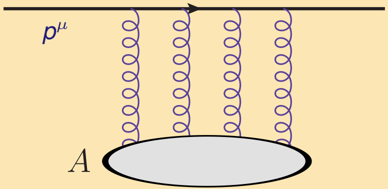
**Color Glass Condensate (CGC)**

We assume that there are so many gluons in the target, that it can be described by a classical gluon field. This is the heart of the CGC effective theory.

Many gluons = large color field $A_\mu$
Have to sum all diagrams with $n$ gluons lines
— but we can assume the gluons are a classical field



# What is the target made of?




Quark propagating in classical color field: Dirac equation!

$$(i\partial\!\!\!/ - gA\!\!\!/)\psi(x) = 0$$

(Note: $A\!\!\!/ = A_a^\mu \gamma_\mu t^a$ is $N_c \times N_c$-matrix )

Want to dig out the dominant contribution: **eikonal** approximation

- Gluon is spin 1: it couples to a vector: $\sim p^\mu A_\mu$
- For high energy particle the only momentum available is $p^\mu$
- $p^\mu$ has one large component: $p^+ \implies p^\mu A_\mu \sim p^+ A^- \implies$ only need $A^-$

Ansatz for DE: $\psi(x) = V(x)e^{-ip\cdot x}u(p)$, plug into equation **$N_c \times N_c$-matrix!**

$$\implies \quad \partial_+ V(x^+, x^-, \mathbf{x}) = -igA^-(x^+, x^-, \mathbf{x}) V(x^+, x^-, \mathbf{x})$$

**This is solved by path-ordered exponential**

$$V(x^+, x^-, \mathbf{x}) = \mathbb{P}\exp\left\{-ig\int^{x^+} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x})\right\}$$

# Eikonal propagation

- Now we know how a high energy quark propagates in a classical field.
- Thus we know the scattering $S$-matrix element for many-quark states

E.g. incoming free quark $|q_i(\mathbf{x})\rangle$ at $x^+ \to -\infty$ is, at $x^+ \to \infty$

$$|q_i(\mathbf{x})\rangle_{\text{in}} = \left[\mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x})\right\}\right]_{ji} |q_j(\mathbf{x})\rangle_{\text{out}}$$

a linear superposition of color rotated outgoing quarks.

- In scattering problem integrate $x^+ \in [-\infty, \infty]$
- In the high energy limit quark wavefunction oscillates like $e^{ip^+x^-}$ with large $p^+$ $\implies$ $x^-$-dependence negligible compared to this $\implies$ approximate $x^- = 0$

**Scattering is described by 2-dimensional field of SU($N_\mathrm{c}$)-matrices**

$$V(\mathbf{x}) \equiv \mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}x^+ A^-(x^+, x^- = 0, \mathbf{x})\right\}$$

— These is known as the **Wilson lines**

# Dipole amplitude and Wilson lines

Incoming dipole (color neutral, average over colors!) changes into

$$|\mathrm{in}\rangle = \frac{\delta_{ii'}}{N_c}\,|q_i(\mathbf{x})\bar{q}_{i'}(\mathbf{y})\rangle_{\mathrm{in}} = \frac{\delta_{ii'}}{N_c} V_{ji}(\mathbf{x})V^{\dagger}_{i'j'}(\mathbf{y})\,|q(\mathbf{x})_j\bar{q}(\mathbf{y})_{j'}\rangle_{\mathrm{out}} \qquad (V(\mathbf{y})^{\dagger}_{jk} = V(\mathbf{y})^{*}_{kj} \text{ for antiquark})$$

The total cross section is related to the imaginary part of the **forward elastic scattering amplitude**; i.e. we need to count outgoing dipoles in this state

$$S = {}_{\mathrm{out}}\langle q_k(\mathbf{x})\bar{q}_k(\mathbf{y})|\,\mathrm{in}\rangle = \frac{\delta_{ii'}}{N_c}\delta_{kj}\delta_{kj'}\,V_{ji}(\mathbf{x})V^{\dagger}_{i'j'}(\mathbf{y}) = \frac{1}{N_c}\,\mathrm{Tr}\,V(\mathbf{x})V^{\dagger}(\mathbf{y})$$

## Dipole amplitude in the CGC

Relate $\mathcal{N}$ in BK and DIS to a **microscopical description of the target**:

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c}\,\mathrm{Tr}\,V(\mathbf{x})V^{\dagger}(\mathbf{y})$$

## Note conventions

$$S_{fi} = \langle f|\,\hat{S}\,|f\rangle = 1 + iT_{fi} \qquad \sigma_{\mathrm{tot}} = 2\,\mathrm{Im}\,T_{ii} \qquad \mathcal{N} \equiv \mathrm{Im}\,T_{ii} \qquad S_{ii} = \delta_{ii} - \mathcal{N} + \mathrm{imag}$$

## More complicated operators

- The dipole amplitude is a target expectation value of a two-point function

$$\mathcal{N}_{q\bar{q}} = 1 - \left\langle \hat{D} \right\rangle = \left\langle 1 - \frac{1}{N_c} \operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y}) \right\rangle_{\text{target}}$$

- For this we derived the BK equation using a **mean field** approximation $\left\langle \hat{D}\hat{D} \right\rangle \approx \left\langle \hat{D} \right\rangle \left\langle \hat{D} \right\rangle$
- Similarly define other correlators, such as $\left\langle \hat{D}\hat{D} \right\rangle$ or the quadrupole

$$Q = \left\langle \frac{1}{N_c} \operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y}) V(\mathbf{u}) V^\dagger(\mathbf{v}) \right\rangle_{\text{target}},$$

  and the corresponding evolution equations.
- Without the mean field approx. these operators couple to each other (e.g. $\partial_y \left\langle \hat{D} \right\rangle \sim \left\langle \hat{D}\hat{D} \right\rangle$) the **Balitsky hierarchy** of evolution equations
- The hierarchy can be generalized into an evolution equation for the **probability distribution of Wilson lines** — the JIMWLK equation

# From BK to JIMWLK

## JIMWLK equation

Gives rapidity-dependence of probability distribution of Wilson lines

$$\partial_y W_y[U(\mathbf{x})] = \mathcal{H} W_y[U(\mathbf{x})]$$

$$\mathcal{H} \equiv \frac{1}{2} \int_{\mathbf{xyz}} \frac{\delta}{\delta \widetilde{A}_c^+(\mathbf{y})} \mathbf{e}^{ba}(\mathbf{x},\mathbf{z}) \cdot \mathbf{e}^{ca}(\mathbf{y},\mathbf{z}) \frac{\delta}{\delta \widetilde{A}_b^+(\mathbf{x})},$$

$$\mathbf{e}^{ba}(\mathbf{x},\mathbf{z}) = \frac{1}{\sqrt{4\pi^3}} \frac{\mathbf{x}-\mathbf{z}}{(\mathbf{x}-\mathbf{z})^2} \left(1 - U^\dagger(\mathbf{x}) U(\mathbf{z})\right)^{ba}$$

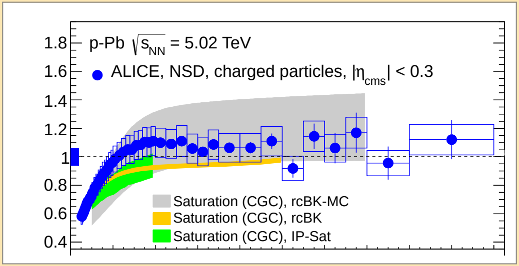
You can derive this in a very similar way as we did for BK.

- Assume there is a $y$-dependent probability distribution $W_y[U(\mathbf{x})]$
- Consider collection of $n$ Wilson lines propagating through target
- Emit one extra soft gluon and absorb small-$z$ divergence into redefinition of probability distribution: $W_y[U(\mathbf{x})] \to W_{y+\Delta y}[U(\mathbf{x})]$

# 4 Particle production in pA

# Nuclear modification factor $R_{pA}$

Comparison of ALICE data on particle production in pA and pp to some theory predictions

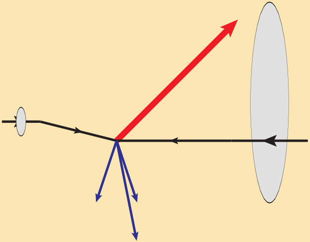



**There are two ways to calculate this in the CGC**

*$k_T$-factorization* Good at midrapidity/symmetric situation with strong color fields in **both** colliding objects. This we will come to a bit later

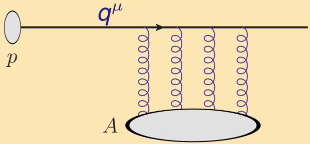
*Hybrid formalism* One colliding object described as dilute collection of partons $\Longrightarrow$ good at forward rapidity. Let us first understand this

# Dilute-dense scattering



Look at forward rapidity pA

- The produced particle has large $p^+$.
- Momentum conservation: it comes from large $x$ parton in proton
- At large $x$ the proton is dilute collection of valence quarks
  $\Longrightarrow$ quark scattering on dense target




In: quark with momentum $q^+$, $\mathbf{q}$, color $i$

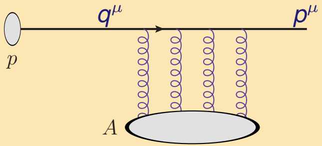
$$|\mathrm{in}\rangle = \int \mathrm{d}^2\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} |q_i(\mathbf{x})\rangle_{\mathrm{in}}$$

After interaction with the target

$$|\mathrm{in}\rangle = \int \mathrm{d}^2\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} V_{ji}(\mathbf{x}) |q(\mathbf{x})_j\rangle_{\mathrm{out}}$$

## Scattering amplitude

$$|\mathrm{in}\rangle = \int \mathrm{d}^2\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} V_{ji}(\mathbf{x}) \, |q(\mathbf{x})_j\rangle_{\mathrm{out}}$$




Scattering amplitude by projecting quarks with momentum **p** in the final state

(Cheating and forgetting the 1 in $S = 1 + iT$)

$$\mathcal{M}_{i,\mathbf{q}\to k,\mathbf{p}} = {}_{\mathrm{out}}\langle q_k(\mathbf{p})| \, \mathrm{in}\rangle = \int \mathrm{d}^2\mathbf{x}\, \mathrm{d}^2\mathbf{y} e^{-i(\mathbf{q}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} V_{ji}(\mathbf{x})\, \overbrace{{}_{\mathrm{out}}\langle q_k(\mathbf{y})| \, q(\mathbf{x})_j\rangle_{\mathrm{out}}}^{\delta^2(\mathbf{y}-\mathbf{x})\delta_{kj}}$$

We can choose $\mathbf{q} = 0$

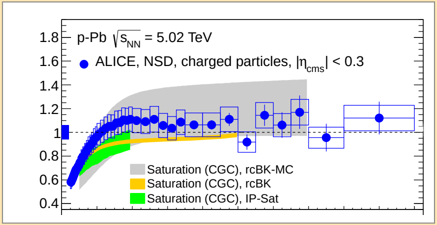
$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\mathbf{p}} = \frac{1}{N_\mathrm{c}} \frac{1}{(2\pi)^2} \sum_{i,k} |\mathcal{M}_{i,\mathbf{q}\to k,\mathbf{p}}|^2 = \frac{1}{N_\mathrm{c}} \frac{1}{(2\pi)^2} \int \mathrm{d}^2\mathbf{x}\, \mathrm{d}^2\mathbf{y} e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y})$$

There are $xq(x,\mu^2)$ incoming quarks in the proton per unit rapidity.

**Hybrid formula for quark production**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\mathbf{p}\,\mathrm{d}y} = \frac{1}{(2\pi)^2} xq(x,\mu^2) \frac{1}{N_\mathrm{c}} \int \mathrm{d}^2\mathbf{x}\, \mathrm{d}^2\mathbf{y} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y})$$

# Back to $R_{pA}$




$$\frac{d\sigma}{d^2\mathbf{q}\,dy} = \frac{1}{(2\pi)^2} xq(x,\mu^2) \frac{1}{N_c} \int d^2\mathbf{x}\, d^2\mathbf{y} e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y})$$

Now all we need is a parametrization, for protons **and** nuclei of

$$\operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y})$$

- Fit to HERA data $\Longrightarrow$ proton dipole amplitude
  - using BK equation (remember: BK gives $x$-dependence, need to fit initial condition)
  - or some other model of the dipole cross section
- Generalize to nuclei: somehow incorporate Woods-Saxon $T_A(b)$
- The HERA data is very precise and theory fits it well: the "theory errors" in the above plot are all from this proton $\Longrightarrow$ nucleus generalization.

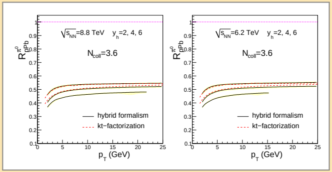
# From protons to nuclei

One typical initial condition for BK: GBW Golec-Biernat, Wusthoff:

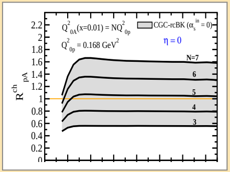
$$\mathcal{N}(\mathbf{b},\mathbf{r}) = \theta(R_p - b)\left(1 - \exp\left\{-\frac{\mathbf{r}^2}{4Q_s^2}\right\}\right), \qquad \text{and for nucleus?}$$

1. Just fit $Q_s^A$ separately to some nuclear data
2. Assume saturation scale $Q_s^2 \sim T_A(\mathbf{b})$ or $A^{1/3}$ — with what coefficient?
3. MC Glauber, count overlapping nucleons and $\left(Q_s^A\right)^2 = N_N \left(Q_s^A\right)^2$
   — Fine, but what is the area of the nucleon when you calculate $N_N$? Same as in DIS? Same as in Glauber? (These are different!)

One has to be careful (I'm being nasty showing these celebrated plots)




Oops!



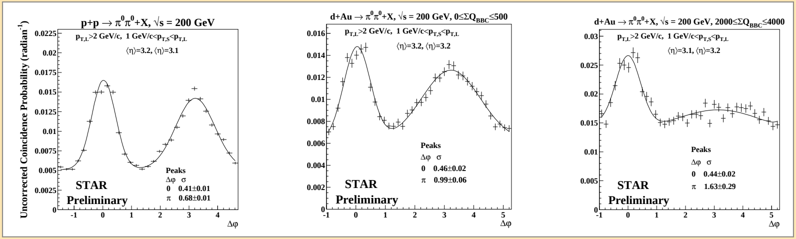

And the prediction was?

Differences mostly in nuclear geometry, not in the QCD!

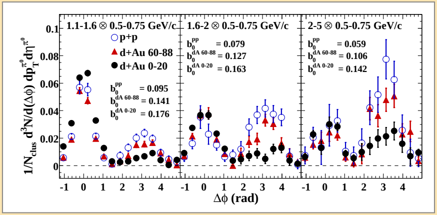
# Another example: forward dihadron correlations in dAu

Two particle collision vs. $\Delta\varphi$ :

pp peripheral *d*Au central *d*Au



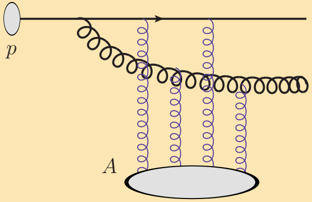

STAR, [arXiv: 1102.0931]




PHENIX, [arXiv: 1105.5112], PRL

# Calculating 2-particle correlation in forward pA

- Quark from $p$ (large $x$) from pdf, radiate gluon
- Propagate eikonally through target $\Longrightarrow$ Wilson lines $U(\mathbf{x})$
- Need target expectation values of Wilson lines — from JIMWLK




$$\frac{\mathrm{d}\sigma^{qA\to qgX}}{\mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{k}} \propto \int_{\mathbf{x},\bar{\mathbf{x}},\mathbf{y},\bar{\mathbf{y}}} e^{-i\mathbf{q}\cdot(\mathbf{x}-\bar{\mathbf{x}})}\, e^{-i\mathbf{k}\cdot(\mathbf{y}-\bar{\mathbf{y}})}\, \mathcal{F}(\bar{\mathbf{x}}-\bar{\mathbf{y}},\mathbf{x}-\mathbf{y})$$

$$\left\langle \hat{Q}(\mathbf{y},\bar{\mathbf{y}},\bar{\mathbf{x}},\mathbf{x})\,\hat{D}(\mathbf{x},\bar{\mathbf{x}}) - \hat{D}(\mathbf{y},\mathbf{x})\hat{D}(\mathbf{x},\bar{\mathbf{z}}) - \hat{D}(\mathbf{z},\bar{\mathbf{x}})\hat{D}(\bar{\mathbf{x}},\bar{\mathbf{y}}) + \dots \right\rangle_{\text{target}}$$

$(\mathbf{z} = z\mathbf{x} + (1-z)\mathbf{y},\ \bar{\mathbf{z}} = z\bar{\mathbf{x}} + (1-z)\bar{\mathbf{y}}.)$

$$\hat{D}(\mathbf{x}-\mathbf{y}) \equiv \frac{1}{N_c}\operatorname{Tr} U(\mathbf{x})U^\dagger(\mathbf{y}) \quad \hat{Q}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) \equiv \frac{1}{N_c}\operatorname{Tr} U(\mathbf{x})U^\dagger(\mathbf{y})U(\mathbf{u})U^\dagger(\mathbf{v})$$

# 5 Gluon saturation and the CGC

# Classical field and equation of motion

- We were describing the high energy nucleus as a classical field: $A^-$ $\Longrightarrow$ Wilson line
- What does this imply for the partonic content of the nucleus?
- The physical picture of "gluons as partons" requires two things (cf. Marco's lectures)
  - Infinite momentum frame: nucleus moving fast
    Also change direction: nucleus moves now in $+z$-direction with large $p^+$.
    Means we have large $A^+$ component.
  - Light cone gauge: have to gauge transform to $A^+ = 0$
- But let us start with the "classical" part.

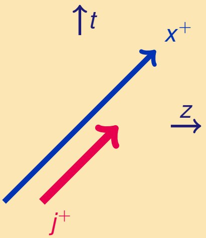
Classical field $\equiv$ from equation of motion

$$[D_\mu, F^{\mu\nu}] = J^\mu$$

What remains is

$$\nabla^2 A^+ = J^+$$

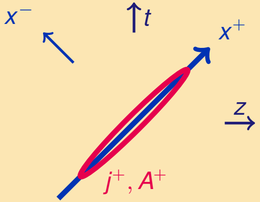
This is nice, the big $+$-field corresponds to a **color** current in the $+$-direction.



# Spacetime structure of the field

The current lives on the light cone.

1. Naive explanation: Nucleus is Lorentz-contracted to $\Delta z \sim 2R_A m_A/\sqrt{s}$
2. Real explanation: Current represents large $x$ degrees of freedom
   - They have large $p^+$, classical field small
   - They are more localised in $x^-$ than the field.




Extreme approximation:

$$
\begin{aligned}
j^+(x^-,\mathbf{x}) &\approx \delta(x^-)\rho(\mathbf{x}) \\
A^+(x^-,\mathbf{x}) &\approx \delta(x^-)\frac{1}{\boldsymbol{\nabla}^2}\rho(\mathbf{x})
\end{aligned}
$$

---

The current is independent of LC time $x^+$;

**glass!**

Argument is as above:

1. Time is dilated for the nucleus
2. Any probe will have larger $k^-$ than color current $\Longrightarrow$ probe will oscillate faster in $x^+$ and see current as static.

# Classical field and equation of motion

Now let us gauge transform.

$$A^+ \Rightarrow U^\dagger(\mathbf{x},x^-)A^+U(\mathbf{x},x^-) - \frac{i}{g}U^\dagger(\mathbf{x},x^-)\partial_- U(\mathbf{x},x^-) = 0$$

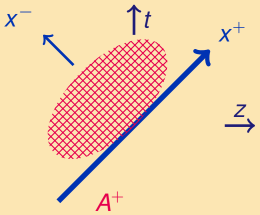
$$A^- \Rightarrow -\frac{i}{g}U^\dagger(\mathbf{x},x^-)\partial_+ U(\mathbf{x},x^-) = 0, \text{ still}$$

$$A^i \Rightarrow \frac{i}{g}U^\dagger(\mathbf{x},x^-)\partial_i U(\mathbf{x},x^-)$$

**transverse pure gauge**

This is solved by familiar Wilson line

$$U(\mathbf{x},x^-) = \mathbb{P}\exp\left[-ig\int^{x^-} \mathrm{d}y^- A^+\right]$$




Now $A^i \sim \theta(x^-)$ — delocalized in $x^-$, just like small $k^+$ physical gluons should be.

# Weizsäcker-Williams gluon distribution

In LC quantization (Now of nucleus, not $\gamma^*$) the number distribution of gluons:

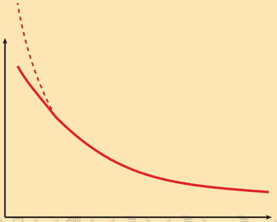
$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} \sim \left\langle A_a^i(\mathbf{k}) A_a^i(-\mathbf{k}) \right\rangle$$

- $A_a^i(\mathbf{k})$ is obtained from the Wilson line
- Wilson line is related to DIS dipole cross section, BK equation
- One can express this **Weizsäcker-Williams** gluon distribution as:

$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = \varphi^{\mathrm{WW}}(\mathbf{k}) = \frac{C_\mathrm{F}}{2\pi^3}\frac{1}{\alpha_\mathrm{s}} \int \mathrm{d}^2\mathbf{b} \int \mathrm{d}^2\mathbf{r}\, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{r}^2} \widetilde{\mathcal{N}}(\mathbf{b},\mathbf{r})$$

($\widetilde{\mathcal{N}}$ is the adjoint representation Wilson line correlator)

- You can write the dipole formula for DIS in a $k_T$-factorized form that involves $\varphi^{\mathrm{WW}}(\mathbf{k})$
- Gluon saturation in $\varphi^{\mathrm{WW}}(\mathbf{k})$ at $\mathbf{k} \lesssim Q_\mathrm{s}$
- $\varphi^{\mathrm{WW}}(\mathbf{k}) \sim 1/\alpha_\mathrm{s} \implies$ "**condensate**" of gluons



Now we have a **C**olor **G**lass **C**ondensate.

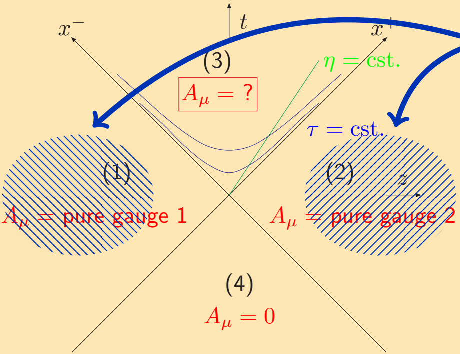
# 6 Heavy ion collisions and the glasma initial state

# Gluon fields in AA collision

Now two colliding nuclei $\Longrightarrow$ two color currents

$$J^\mu = \delta^{\mu +}\rho_{(1)}(\mathbf{x})\delta(x^-) + \delta^{\mu -}\rho_{(2)}(\mathbf{x})\delta(x^+)$$

**C**lassical **Y**ang-**M**ills




2 pure gauges

$$A^i_{(1,2)} = \frac{i}{g} U_{(1,2)}(\mathbf{x})\partial_i U^\dagger_{(1,2)}(\mathbf{x})$$

$$U_{(1,2)}(\mathbf{x}) = P e^{ig\int dx^- \frac{\rho(\mathbf{x},x^-)}{\nabla^2}}$$

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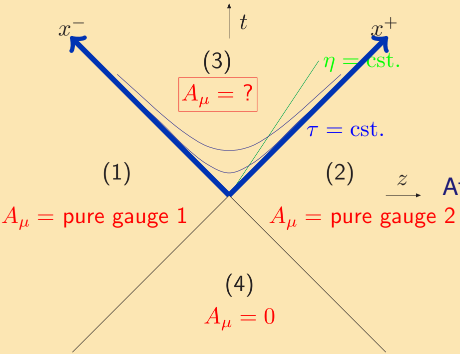


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$$U_{(1,2)}(\mathbf{x}) = P e^{ig\int dx^- \frac{\rho(\mathbf{x},x^-)}{\nabla^2}}$$

At $\tau = 0$:

$$A^i\Big|_{\tau=0} = A^i_{(1)} + A^i_{(2)}$$

$$A^\eta\Big|_{\tau=0} = \frac{ig}{2}[A^i_{(1)}, A^i_{(2)}]$$

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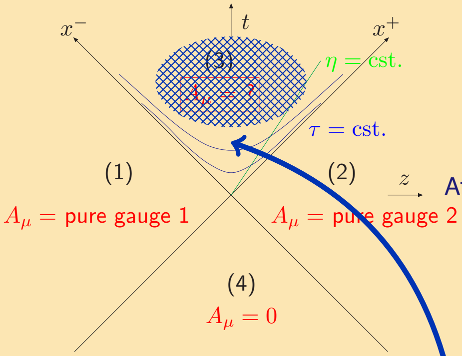


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At $\tau = 0$:

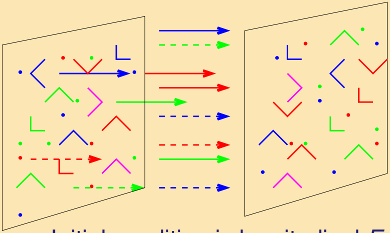
$$A^i\Big|_{\tau=0} = A^i_{(1)} + A^i_{(2)}$$

$$A^\eta\Big|_{\tau=0} = \frac{ig}{2}[A^i_{(1)}, A^i_{(2)}]$$

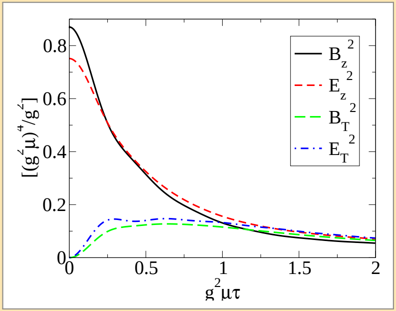
Solve numerically Yang-Mills equations for $\boldsymbol{\tau > 0}$

This is the **glasma** field $\Longrightarrow$ Then average over $\rho$.

# Result: glasma field



- Initial condition is longitudinal $E$ and $B$ field,
- Depend on transverse coordinate with correlation length $1/Q_s$.
  $\Longrightarrow$ gluon correlations




Gauss law and Bianchi: (here $i = 1 \ldots 3$)

$$[D_i, E^i] = 0, \quad [D_i, B^i] = 0$$

Separate nonabelian parts:

$$\partial_i E^i = ig[A^i, E^i], \quad \partial_i B^i = ig[A^i, B^i]$$

Effective electric and magnetic charge densities.

# Deriving the initial condition

Let's work in Fock-Schwinger/temporal gauge $A_\tau = \left(x^+ A^- + x^- A^-\right)/\tau = 0$
$\Longrightarrow$ consistent with LC gauge solutions for both nuclei.

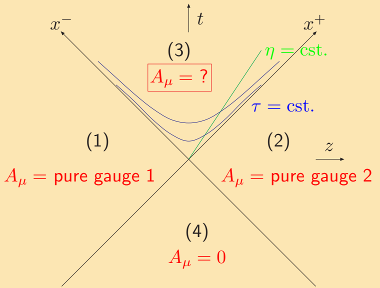
$$\begin{aligned}
\text{Ansatz: } A_i &= \overbrace{A_i^{(1)}\theta(-x^+)\theta(x^-) + A_i^{(2)}\theta(x^+)\theta(-x^-)}^{\text{known}} + A_i^{(3)}\theta(x^+)\theta(x^-) \\
A^\pm &= \pm\theta(x^+)\theta(x^-)x^\pm A^\eta
\end{aligned}$$

Insert into $[D_\mu, F^{\mu\nu}]$ and match $\delta$-functions

▼

initial condition for region (3):

$$\begin{aligned}
A_i^{(3)}|_{\tau=0} &= A_i^{(1)} + A_i^{(2)} \\
A^\eta|_{\tau=0} &= \frac{ig}{2}\left[A_i^{(1)}, A_i^{(2)}\right]
\end{aligned}$$



# Gluon spectrum in the glasma

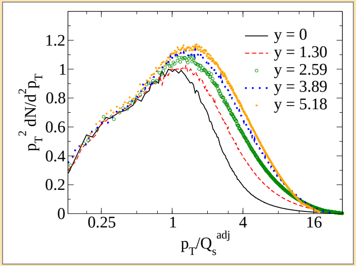
CYM equations can be solved numerically on the lattice.
Decompose solution in Fourier **k**-modes: gluon spectrum

**$Q_s$ is only dominant scale**

Parametrically $$\frac{dN_g}{dy\, d^2\mathbf{x}\, d^2\mathbf{p}} = \frac{1}{\alpha_s} f\left(\frac{p}{Q_s}\right)$$

Produced gluon spectrum: harder at higher $\sqrt{s}$

(Here: midrapidity, $y \equiv \ln \sqrt{s/s_0}$)



# Dilute limit and $k_T$-factorization

The equations of motion are easy to solve in the **dilute limit**;
(This is a CGC theorist's "pp collision")
Linearized eqations are wave equations

$$\begin{aligned}\left(\tau^2\partial_\tau^2 + \tau\partial_\tau + \tau^2\mathbf{k}^2\right) A_i(\tau,\mathbf{k}) &= 0\\ \left(\tau^2\partial_\tau^2 - \tau\partial_\tau + \tau^2\mathbf{k}^2\right) A_\eta(\tau,\mathbf{k}) &= 0.\end{aligned}$$

$$\Longrightarrow \quad A_i(\tau,\mathbf{k}) = A_i(\tau=0,\mathbf{k})J_0(|\mathbf{k}|\tau) \qquad A^\eta(\tau,\mathbf{k}) = -\frac{1}{\tau|\mathbf{k}|}A^\eta(\tau=0,\mathbf{k})J_1(|\mathbf{k}|\tau).$$

- These are (boost invariant) plane waves $\Longrightarrow$ interpret as particles, gluons.
- Initial fields related to Wilson lines, and via that to the gluon amplitude

Number spectrum **in the dilute limit**: $k_T$-factorization formula.

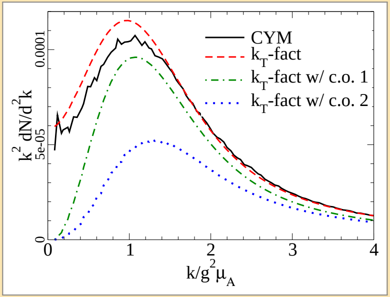
$$\frac{\mathrm{d}N}{\mathrm{d}y\,\mathrm{d}^2\mathbf{k}} = \frac{\alpha_s}{S_\perp}\frac{2}{C_F}\frac{1}{k^2}\int \mathrm{d}^2\mathbf{q}\,\varphi^{\mathrm{dip}}(\mathbf{q})\varphi^{\mathrm{dip}}(|\mathbf{k}-\mathbf{q}|).$$

This calculation can also be repeated by assuming that **one** of the two colliding objects is dilute (Theorist's "pA") — **It does not work in "AA"**

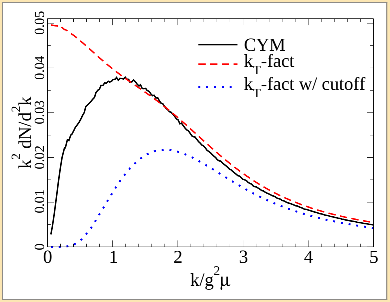
# CYM vs. **k**-factorization

- In fact, also in "AA" the $k_T$-factorization formula works for high $p_T$
- But it does not give a finite **integrated** total gluon multiplicity,
  - Sometimes this is fixed by an ad hoc cutoff

$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{p}\,\mathrm{d}y} = \frac{1}{\alpha_s}\frac{1}{\mathbf{p}^2}\int_{\mathbf{k}} \left[\theta(p-k)\right] \phi_y(\mathbf{k})\phi_y(\mathbf{p}-\mathbf{k})$$

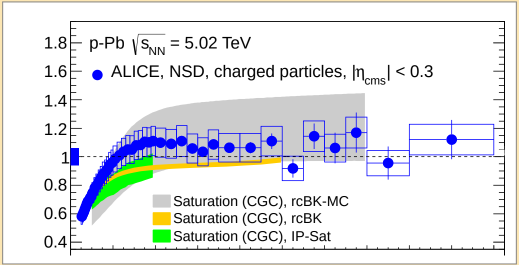



pA: **k**-factorization works




**AA: $k_T$-factorization only for large $p_T$**

# Back to $R_{pA}$




The theory predictions here are calculated with the $k_T$-factorization formula:

$$\frac{\mathrm{d}N}{\mathrm{d}y\,\mathrm{d}^2\mathbf{k}} = \frac{\alpha_\mathrm{s}}{S_\perp}\frac{2}{C_\mathrm{F}}\frac{1}{k^2}\int \mathrm{d}^2\mathbf{q}\,\varphi^{\mathrm{dip}}(\mathbf{q})\varphi^{\mathrm{dip}}(|\mathbf{k}-\mathbf{q}|),$$

convoluted with a fragmentation function for $g \to$hadrons.

- You can also rederive the hybrid formula from this, in the asymmetric limit. $(Q_\mathrm{s}^A \gg Q_\mathrm{s}^p$, i.. $|\mathbf{k}-\mathbf{q}| \gg |\mathbf{q}|)$

# Tale of two gluon distributions

This picture has only been clarified recently. One must differentiate

## WW distribution

$$\varphi^{\mathrm{WW}}(\mathbf{k}) = \frac{C_\mathrm{F}}{2\pi^3}\frac{1}{\alpha_\mathrm{s}}\int \mathrm{d}^2\mathbf{b}\int \mathrm{d}^2\mathbf{r} \times \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{r}^2}\mathcal{N}(\mathbf{b},\mathbf{r})$$

- Comes from actually counting gluons in the nucleus
- Appears in $k_T$-factorized expression for DIS
- Satisfies the usual momentum-space version of the BK equation

## Dipole distribution distribution

$$\varphi^{\mathrm{dip}}(\mathbf{k}) = \frac{C_\mathrm{F}}{8\pi^3}\frac{\mathbf{k}^2}{\alpha_\mathrm{s}}\int \mathrm{d}^2\mathbf{b}\int \mathrm{d}^2\mathbf{r} \times e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{N}(\mathbf{b},\mathbf{r})$$

- Appears in $k_T$-factorized expression for particle production in pp, pA